Nonlinear approximation of DSGE models with Dynare.

Workshop: Identification analysis and global sensitivity analysis for macroeconomic models

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1. computes the solution of deterministic models (arbitrary accuracy),
2. computes first, second and third order approximation to solution of stochastic models,
The general problem

Deterministic, perfect foresight, case:

\[ f(y_{t+1}, y_t, y_{t-1}, u_t) = 0 \]

Stochastic case:

\[ E_t \{ f(y_{t+1}, y_t, y_{t-1}, u_t) \} = 0 \]

\( y \) : vector of endogenous variables

\( u \) : vector of exogenous shocks
Solution methods

- For a deterministic, perfect foresight, it is possible to compute numerical trajectories for the endogenous variables.
- In a stochastic framework, the unknowns are the decision functions:

\[ y_t = g(y_{t-1}, u_t) \]

For a large class of DSGE models, DYNARE computes approximated decision rules and transition equations by a perturbation method.
Example: Neoclassical growth model

\[
\max_{\{c_t\}} \sum_{t=1}^{\infty} \beta^{t-1} \frac{c_t^{1-\sigma}}{1-\sigma}
\]

s.t.

\[c_t + k_t = A_t k_{t-1}^\alpha + (1 - \delta) k_{t-1}\]

First order conditions:

\[c_t^{-\sigma} = \beta c_{t+1}^{-\sigma} \left( \alpha A_{t+1} k_{t-1}^{\alpha-1} + 1 - \delta \right)\]

\[c_t + k_t = A_t k_{t-1}^\alpha + (1 - \delta) k_{t-1}\]

Steady state:

\[\bar{k} = \left( \frac{1 - \beta(1 - \delta)}{\beta \alpha A} \right)^{\frac{1}{\alpha-1}}\]

\[\bar{c} = \bar{A} \bar{k}^\alpha - \delta \bar{k}\]
Calibration

\[ \alpha = 0.3 \]
\[ \beta = 0.98 \]
\[ \delta = 0.025 \]
\[ \sigma = 1 \]
// variables declaration
var c k;
varexo A;

// parameters declaration
parameters alpha beta delta sigma;
alpha = 0.3;
beta = 0.98;
delta = 0.025;
sigma = 1;

// model equations
model;
c^(-sigma) = beta*c(+1)^(-sigma) *
  (alpha*A(+1)*k^(alpha-1)+1-delta);
c+k = A*k(-1)^alpha+(1-delta)*k(-1);
end;
// setting value for exogenous variable
// providing exact value for steady state of
// endogenous variables
steady_state_model;
k = ((1-beta*(1-delta))/
    (beta*alpha*A))^(1/(alpha-1));
c = A*k^alpha-delta*k;
end;
Computation of first order approximation

- Perturbation approach: recovering a Taylor expansion of the solution function from a Taylor expansion of the original model.
- A first order approximation is nothing else than a standard solution thru linearization.
- A first order approximation in terms of the logarithm of the variables provides standard log-linearization.
General model

\[ E_t \{ f(y_{t+1}, y_t, y_{t-1}, u_t) \} = 0 \]

\[ E(u_t) = 0 \]
\[ E(u_t u'_t) = \Sigma_u \]
\[ E(u_t u'_\tau) = 0 \quad t \neq \tau \]

\( y \) : vector of endogenous variables

\( u \) : vector of exogenous stochastic shocks
Timing assumptions

\[ E_t \{ f(y_{t+1}, y_t, y_{t-1}, u_t) \} = 0 \]

- shocks \( u_t \) are observed at the beginning of period \( t \),
- decisions affecting the current value of the variables \( y_t \), are function of
  - the previous state of the system, \( y_{t-1} \),
  - the shocks \( u_t \).
The stochastic scale variable

\[ E_t \left\{ f(y_{t+1}, y_t, y_{t-1}, u_t) \right\} = 0 \]

- At period \( t \), the only unknown stochastic variable is \( y_{t+1} \), and, implicitly, \( u_{t+1} \).
- We introduce the stochastic scale variable, \( \sigma \) and the auxiliary random variable, \( \epsilon_t \), such that

\[ u_{t+1} = \sigma \epsilon_{t+1} \]
The stochastic scale variable (continued)

\[ E(\epsilon_t) = 0 \quad (1) \]
\[ E(\epsilon_t \epsilon'_t) = \Sigma_{\epsilon} \quad (2) \]
\[ E(\epsilon_t \epsilon'_\tau) = 0 \quad t \neq \tau \quad (3) \]

and

\[ \Sigma_u = \sigma^2 \Sigma_{\epsilon} \]
Remarks

\[ E_t \{ f(y_{t+1}, y_t, y_{t-1}, u_t) \} = 0 \]

- The exogenous shocks may appear only at the current period
- There is no deterministic exogenous variables
- Not all variables are necessarily present with a lead and a lag
- Generalization to leads and lags on more than one period (2nd order approximation requires a more complicated algorithm)
Solution function

\[ y_t = g(y_{t-1}, u_t, \sigma) \]

where \( \sigma \) is the stochastic scale of the model. If \( \sigma = 0 \), the model is deterministic. For \( \sigma > 0 \), the model is stochastic. Under some conditions, the existence of \( g() \) function is proven via an implicit function theorem. See H. Jin and K. Judd “Solving Dynamic Stochastic Models”

(http://bucky.stanford.edu/papers/PerturbationMethodRatEx.pdf)
Solution function (continued)

Then,

\[
y_{t+1} = g(y_t, u_{t+1}, \sigma) \\
= g(g(y_{t-1}, u_t, \sigma), u_{t+1}, \sigma)
\]

\[
F(y_{t-1}, u_t, \epsilon_{t+1}, \sigma) \\
= f(g(g(y_{t-1}, u_t, \sigma), \sigma\epsilon_{t+1}, \sigma), g(y_{t-1}, u_t, \sigma), y_{t-1}, u_t)
\]

\[
E_t \{ F(y_{t-1}, u_t, \epsilon_{t+1}, \sigma) \} = 0
\]
The perturbation approach

- Obtain a Taylor expansion of the unknown solution function in the neighborhood of a problem that we know how to solve.
- The problem that we know how to solve is the deterministic steady state.
- One obtains the Taylor expansion of the solution for the Taylor expansion of the original problem.
- One consider two different perturbations:
  1. points in the neighborhood from the steady state,
  2. from a deterministic model towards a stochastic one (by increasing $\sigma$ from a zero value).
The Taylor approximation is taken with respect to $y_{t-1}$, $u_t$ and $\sigma$, the arguments of the solution function

$$y_t = g(y_{t-1}, u_t, \sigma).$$

At the deterministic steady state, all derivatives are deterministic as well.
Steady state

A deterministic steady state, $\bar{y}$, for the model satisfies

$$f(\bar{y}, \bar{y}, \bar{y}, 0) = 0$$

A model can have several steady states, but only one of them will be used for approximation. Furthermore,

$$\bar{y} = g(\bar{y}, 0, 0)$$
First order approximated decision function

\[ y_t = \bar{y} + g_y \hat{y} + g_u u \]

\[ E \{ y_t \} = \bar{y} \]
\[ \Sigma_y = g_y \Sigma_y g_y' + \sigma^2 g_u \Sigma_\epsilon g_u' \]

The variance is solved for with an algorithm for Lyapunov equations.
Second and third order approximation of the model

- Second and third order approximation of the solution function are obtained from second, respectively third, order approximation of the model.
- It requires only the solution of (tricky) linear problems.
- The stochastic scale of the model, $\sigma$, appears in the solution and breaks certainty equivalence.
Second order approximation of the model

\[ E_t \left\{ F^{(2)}(y_{t-1}, u_t, \epsilon_{t+1}, \sigma) \right\} = \\
E_t \left\{ F^{(1)}(y_{t-1}, u_t, u_{t+1}, \sigma) \right\} \\
+ 0.5 \left( F_{y_y} (\hat{y} \otimes \hat{y}) + F_{uu} (u \otimes u) + F_{u'u'} \sigma^2 (\epsilon' \otimes \epsilon') + F_{\sigma\sigma} \sigma^2 \right) \\
+ F_{y_u} (\hat{y} \otimes u) + F_{y_u'} (\hat{y} \otimes \sigma \epsilon') + F_{y_\sigma} \hat{y} \sigma + F_{u'u'} (u \otimes \sigma \epsilon') + F_{u\sigma} u \sigma + F_{u'u \sigma} \sigma \epsilon' \sigma \right\} \\
= E_t \left\{ F^{(1)}(y_{t-1}, u_t, \epsilon_{t+1}, \sigma) \right\} \\
+ 0.5 \left( F_{y_y} (\hat{y} \otimes \hat{y}) + F_{uu} (u \otimes u) + F_{u'u'} (\sigma^2 \Sigma_{\epsilon}) + F_{\sigma\sigma} \sigma^2 \right) \\
+ F_{y_u} (\hat{y} \otimes u) + F_{y_\sigma} \hat{y} \sigma + F_{u\sigma} u \sigma \\
= 0 \]
Representing the second order derivatives

The second order derivatives of a vector of multivariate functions is a three dimensional object. We use the following notation

$$\frac{\partial^2 F}{\partial x \partial x} = \begin{bmatrix} \frac{\partial^2 F_1}{\partial x_1 \partial x_1} & \frac{\partial^2 F_1}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 F_1}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 F_1}{\partial x_n \partial x_n} \\ \frac{\partial^2 F_2}{\partial x_1 \partial x_1} & \frac{\partial^2 F_2}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 F_2}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 F_2}{\partial x_n \partial x_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial^2 F_m}{\partial x_1 \partial x_1} & \frac{\partial^2 F_m}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 F_m}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 F_m}{\partial x_n \partial x_n} \end{bmatrix}$$
Composition of two functions

Let

\[ y = g(s) \]
\[ f(y) = f(g(s)) \]

then,

\[ \frac{\partial^2 f}{\partial s \partial s} = \frac{\partial f}{\partial y} \frac{\partial^2 g}{\partial s \partial s} + \frac{\partial^2 f}{\partial y \partial y} \left( \frac{\partial g}{\partial s} \otimes \frac{\partial g}{\partial s} \right) \]
Recovering $g_{yy}$

\[
F_{y-y} = f_{y+} (g_{yy}(g_y \otimes g_y) + g_y g_{yy}) + f_{y_0} g_{yy} + B \\
= 0
\]

where $B$ is a term that doesn’t contain second order derivatives of $g()$.

The equation can be rearranged:

\[
(f_{y+} g_y + f_{y_0}) g_{yy} + f_{y+} g_{yy} (g_y \otimes g_y) = -B
\]

This is a Sylvester type of equation and must be solved with an appropriate algorithm.
Recovering $g_{yu}$

$$F_{y-u} = f_{y+} (g_{yy}(g_y \otimes g_u) + g_y g_{yu}) + f_{y_0} g_{yu} + B$$
$$= 0$$

where $B$ is a term that doesn’t contain second order derivatives of $g()$. This is a standard linear problem:

$$g_{yu} = - (f_{y+} g_y + f_{y_0})^{-1} (B + f_{y+} g_{yy}(g_y \otimes g_u))$$
Recovering $g_{uu}$

\[
F_{uu} = f_y (g_y g_u \otimes g_u) + g_y g_{uu} + f_y g_{uu} + B
\]

\[
= 0
\]

where $B$ is a term that doesn’t contain second order derivatives of $g()$.

This is a standard linear problem:

\[
g_{uu} = - (f_y + g_y + f_{y_0})^{-1} (B + f_y g_y g_u \otimes g_u)
\]
Recovering $g_{y\sigma}, g_{u\sigma}$

\[
F_{y\sigma} = f_y g_y g_{y\sigma} + f_{y0} g_{y\sigma} \\
= 0
\]

\[
F_{u\sigma} = f_y g_y g_{u\sigma} + f_{y0} g_{u\sigma} \\
= 0
\]

as $g_\sigma = 0$. Then

\[g_{y\sigma} = g_{u\sigma} = 0\]
Recovering $g_{\sigma\sigma}$

\[
F_{\sigma\sigma} + F_{u'u'}\Sigma_\epsilon = f_{y_+} (g_{\sigma\sigma} + g_y g_{\sigma\sigma}) + f_{y_0} g_{\sigma\sigma} \\
+ (f_{y_+} y_+ (g_u \otimes g_u) + f_{y_+} g_{uu}) \hat{\Sigma}_\epsilon \\
= 0
\]

taking into account $g_{\sigma} = 0$.
This is a standard linear problem:

\[
g_{\sigma\sigma} = - (f_{y_+} (I + g_y) + f_{y_0})^{-1} (f_{y_+} y_+ (g_u \otimes g_u) + f_{y_+} g_{uu}) \hat{\Sigma}_\epsilon
\]
Second and third order decision functions

- **Second order**

\[
y_t = \bar{y} + 0.5g_{\sigma\sigma}\sigma^2 + g_y\hat{y} + g_uu + 0.5 (g_{yy}(\hat{y} \otimes \hat{y}) + g_{uu}(u \otimes u)) + g_{yu}(\hat{y} \otimes u)
\]

- **Third order**

\[
y_t = \bar{y} + \frac{1}{2}g_{\sigma\sigma}\sigma^2 + \frac{1}{6}\sigma^3 + \frac{1}{2}g_{\sigma\rho\sigma}\hat{\sigma}\sigma^2 + \frac{1}{2}g_{u\sigma\sigma}u\sigma^2 \\
+ g_y\hat{y} + g_uu + \frac{1}{2} (g_{yy}(\hat{y} \otimes \hat{y}) + g_{ uu}(u \otimes u)) + g_{yu}(\hat{y} \otimes u) \\
+ \frac{1}{6} (g_{yyy}(\hat{y} \otimes \hat{y} \otimes \hat{y}) + g_{uuu}(u \otimes u \otimes u)) \\
+ \frac{1}{2} (g_{yyu}(\hat{y} \otimes \hat{y} \otimes u) + g_{yuu}(\hat{y} \otimes \hat{y} \otimes u))
\]

We can fix \(\sigma = 1\).
Second order accurate moments

\[ \Sigma_y = g_y \Sigma_y g'_y + \sigma^2 g_u \Sigma \epsilon g'_u \]

\[ E \{ y_t \} = \bar{y} + (I - g_y)^{-1} \left( 0.5 \left( g_{\sigma \sigma} + g_{yy} \bar{\Sigma}_y + g_{uu} \bar{\Sigma}_\epsilon \right) \right) \]
Further issues

- Impulse response functions depend on the state at time of shocks and history of future shocks.
- For large shocks, second-order approximation simulation may explode:
  - pruning algorithm (Sims)
  - truncate normal distribution (Judd)
Dynare example: Neoclassical growth model (I)

// variables declaration
var c k A;
varexo ea;

// parameters declaration
parameters alpha beta delta sigma;
alpha = 0.3;
beta = 0.98;
delta = 0.025;
sigma = 1;

// model equations
model;
c^(-sigma) = beta*c(+1)^(-sigma)
* (alpha*A(+1)*k^(alpha-1)+1-delta);
c+k = A*k(-1)^alpha+(1-delta)*k(-1);
A = exp(ea);
end;
Dynare example: Neoclassical growth model (II)

steady_state_model;
A = 1;
k = ((1-beta*(1-delta))/
   (beta*alpha*A))^(1/(alpha-1));
c = A*k^alpha-delta*k;
end;

shocks;
var ea; stderr 0.01;
end;

// display steady state
steady;

// check BK conditions
check;

stoch_simul(order=1);
Approximated decision function

POLICY AND TRANSITION FUNCTIONS

<table>
<thead>
<tr>
<th>Constant</th>
<th>1.875089</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k(-1)$</td>
<td>0.072144</td>
</tr>
<tr>
<td>$ea$</td>
<td>0.158801</td>
</tr>
</tbody>
</table>

$$c_t \approx 1.875 + 0.072(k_{t-1} - \bar{k}) + 0.159ea_t$$
The general problem

Deterministic, perfect foresight, case:

\[ f(y_{t+1}, y_t, y_{t-1}, u_t) = 0 \]

- \( y \) : vector of endogenous variables
- \( u \) : vector of exogenous shocks
Solution of deterministic models

- based on work of Laffargue, Boucekkine and myself
- recently much accelerated by Mihoubi
- approximation: impose return to equilibrium in finite time instead of asymptotically
- computes the trajectory of the variables numerically
- uses a Newton–type method
- useful to study full implications of non–linearities
Perfect foresight algorithm

Approximation of an infinite horizon model by a finite horizon one.
The stacked system for a simulation on T periods:

\[
\begin{align*}
  f(y_0, y_1, y_2, x_1) &= 0 \\
  f(y_1, y_2, y_3, x_2) &= 0 \\
  &\vdots \\
  f(y_{T-1}, y_T, y_{T+1}, x_T) &= 0
\end{align*}
\]

for \( y_0 \) and \( y_{T+1} \) given, or

\[ F(Y) = 0 \]

where \( Y = [ y'_1 \ y'_2 \ \ldots \ y'_T ]' \).
A Newton approach

- for an initial guess $Y^{(0)}$
- updated solutions $Y^{(k+1)}$ are obtained by solving

$$\left[ \frac{\partial F}{\partial Y} \right] \left( Y^{(k+1)} - Y^{(k)} \right) = -F(Y^{(k)})$$

- until $\| Y^{(k+1)} - Y^{(k)} \| < \epsilon_Y$ and/or $\| F(Y^{(k)}) \| < \epsilon_F$. 
A sparse Jacobian matrix

\[
\frac{\partial F}{\partial Y} = \begin{bmatrix}
B_1 & C_1 \\
A_2 & B_2 & C_2 \\
& \ddots & \ddots & \ddots \\
& & A_t & B_t & C_t \\
& & & \ddots & \ddots & \ddots \\
& & & & A_{T-1} & B_{T-1} & C_{T-1} \\
& & & & & A_T & B_T
\end{bmatrix}
\]

with

\[
A_t = \frac{\partial f(y_{t-1}, y_t, y_{t+1}, x_t)}{\partial y_{t-1}}
\]

\[
B_t = \frac{\partial f(y_{t-1}, y_t, y_{t+1}, x_t)}{\partial y_t}
\]

\[
C_t = \frac{\partial f(y_{t-1}, y_t, y_{t+1}, x_t)}{\partial y_{t+1}}
\]
Example: neoclassical growth model

\[
\max_{\{c_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} \frac{c_t^{1-\sigma}}{1-\sigma}
\]

s.t.

\[
c_t + k_t = A_t k_{t-1}^{\alpha} + (1 - \delta)k_{t-1}
\]

First order conditions:

\[
c_t^{-\sigma} = \beta c_{t+1}^{-\sigma} \left( \alpha A_{t+1} k_{t-1}^{\alpha-1} + 1 - \delta \right)
\]

\[
c_t + k_t = A_t k_{t-1}^{\alpha} + (1 - \delta)k_{t-1}
\]

Steady state:

\[
\bar{k} = \left( \frac{1 - \beta(1 - \delta)}{\beta \alpha \bar{A}} \right)^{\frac{1}{\alpha-1}}
\]

\[
\bar{c} = \bar{A} \bar{k}^\alpha - \delta \bar{k}
\]
Calibration

$$\alpha = 0.3$$
$$\beta = 0.98$$
$$\delta = 0.025$$
$$\sigma = 1$$
$$\bar{A} = 1$$

First example: return to equilibrium when $$k_0 = 0.5\bar{k}$$. 
neoclassical1.mod

var c k;
varexo A;

parameters alpha beta delta sigma;
alpha = 0.3;
beta = 0.98;
delta = 0.025;
sigma = 1;

model;
c^(-sigma) = beta*c(+1)^(-sigma)
    *(alpha*A(+1)*k^(alpha-1)+1-delta);
c+k = A*k(-1)^alpha+(1-delta)*k(-1);
end;
neoclassical1.mod (continued)

steady_state_model;
k = \left( \frac{1-\beta(1-\delta)}{\beta\alpha A} \right) \left( \frac{1}{\alpha-1} \right);
c = A \cdot k^{\alpha} - \delta \cdot k;
end;

initval;
A=1;
end;

steady;

histval;
k(0) = 0.5 \times \left( \frac{1-\beta(1-\delta)}{\beta\alpha A} \right) \left( \frac{1}{\alpha-1} \right);
end;
simul(periods=100);
rplot k;
rplot c;
A temporary shock to TFP

- the economy is at the steady state
- there is an unexpected drop in TFP of 10% at the beginning of period 1
- deterministic shocks are described in shocks block
- See neoclassical2.mod
steady_state_model;
k = ((1-beta*(1-delta))/(beta*alpha*A))
^ (1/(alpha-1));
c = A*k^alpha-delta*k;
end;

initval;
A=1;
end;

steady;

shocks;
var A;
periods 1;
values 0.9;
end;
A period of temporary favorable shocks announced in the future

- the economy is at the steady state
- TFP jumps by 4% in period 4 and grows by 1% during the 4 following periods
- See neoclassical3.mod

shocks;
var A;
periods 4, 5, 6, 7, 8;
values 1.04, 1.05, 1.06, 1.07, 1.08;
end;
A permanent shock

- the economy is at the initial steady state \((A = 1)\)
- in period 1, TFP jumps to 1.05, permanently
- See neoclassical4.mod
A permanent shock (continued)

```plaintext
steady_state_model;
k = ((1-beta*(1-delta))/(beta*alpha*A))^(1/(alpha-1));
c = A*k^alpha-delta*k;
end;

initval;
A=1;
end;

steady;

derval;
A=1.05;
end;

steady;
```
A pre–announced permanent shock

- the economy is at the initial steady state ($A = 1$)
- in period 6, TFP jumps to 1.05, permanently
- one uses shocks to maintain TFP at initial value during the first 5 periods
- See neoclassical5.mod

```
shocks;
var A;
periods 1:5;
values 1;
end;
```
A two–country overlapping generation model


- All agents live for $T$ periods. The first periods are spent working. The last periods are spent in retirement.
- Agents of age $i$ in country $n = h, f$ maximize

$$
\sum_{t=1}^{T-i+1} \left( \frac{1}{1 + \rho_n} \right)^{t-1} \frac{c_{i,n,t}^{1-\sigma}}{1-\sigma}
$$

- During their active life ($T_A$ periods), they supply one unit of labor in each period.
- They face a budget constraint

$$
S_{i,n,t} + c_{i,n,t} = w_t + (1 + r_t) S_{i-1,n,t-1}
$$

- During retirement, agents live out of their savings, with budget constraint

$$
S_{i,n,t} + c_{i,n,t} = w_t + (1 + r_t) S_{i-1,n,t-1}
$$
OLG model (continued)

- Agents enter active life without savings
  
  \[ s_{0,n,t} = 0 \]

- With perfect-foresight, they don’t leave any inheritance
  
  \[ s_{T,n,t} = 0 \]

- There is a common technology worldwide

- Wage rate
  
  \[ w_t = (1 - \alpha)A k_{t-1}^{\alpha} \]

- Rate of return on capital
  
  \[ r_t = \alpha A k_{t-1}^{\alpha-1} \]

- Worldwide resource constraint with population sizes \( \ell_{n,t} \)
  
  \[ k_t (\ell_{h,t+1} + \ell_{f,t+1}) = s_{h,t} \ell_{h,t} + s_{f,t} \ell_{f,t} \]
First order conditions

\[ c_{i,n,t}^{−\sigma} = \frac{1}{1 + \rho n} c_{i+1,n,t+1}^{−\sigma} (1 + r_{t+1}) \quad i = 1, \ldots, T - 1 \]

\[ s_{1,n,t} + c_{1,n,t} = w_t \]

\[ s_{i,n,t} + c_{i,n,t} = w_t + (1 + r_t) s_{i-1,n,t} \quad i = 2, \ldots, T_A \]

\[ s_{i,n,t} + c_{i,n,t} = (1 + r_t) s_{i-1,n,t-1} \quad i = T_A + 1, \ldots, T - 1 \]

\[ c_{T,n,t} = (1 + r_t) s_{T-1,n,t-1} \]

\[ w_t = A (1 - \alpha) k_{t-1}^{\alpha} \]

\[ r_t = \alpha A k_{t-1}^{\alpha - 1} \]

\[ k_t \left( l_{h,t+1} + l_{f,t+1} \right) = s_{h,t} l_{h,t} + s_{f,t} l_{f,t} \]
Example: 2 countries, 2 generations

```plaintext
var w, hc1, hc2, fc1, fc2, r, k, hs, fs;
varexo drho, fl, hl;

parameters sigma hrho frho alpha A;

sigma = .6;
hrho=0.80;
frho=0.80;
alpha=0.2;
A=1;
```
Example: 2 countries, 2 generations (continued)

model;

(hc2(+1)/hc1)^\sigma = (1+r(+1))/(1+hrho+drho);
hs(-1)*(1+r) = hc2;
w = hs + hc1;
(fc2(+1)/fc1)^\sigma = (1+r(+1))/(1+frho);
fs(-1)*(1+r) = fc2;
w = fs + fc1;

w = A*(1-alpha)*k(-1)^\alpha;
r = A*alpha*k(-1)^{(alpha-1)};
k*(hl(+1)+fl(+1)) = hs*hl+fs*fl;
end;
Example: 2 countries, 2 generations (continued)

initval;
drho=0;
fl=1;
hl=1;

w=.28;
hc1=.12;
hc2=.16;
fcl=.12;
fcl=.12;
frc=.16;
frc=.16;
k=0.15;
r=0.6;
hs=0.2;
fs=0.2;
end;

steady;
A permanent increase of the discount rate in the home country:

```plaintext
endval;
drho=0.05;
end;

steady;

simul(periods=20);

rplot hcl;
rplot r;
```
Example with macros

@define countries = ["h", "f"]
@define T = 10
@define Ta = 6

var w r k;
@for c in countries
  @ for t in 1:T
    var @{c}_c_{t}
    @if t < T
      @{c}_s_{t}
      @endif
  @endif
@endfor
@endfor

varexo drho f_l h_l;
parameters sigma h_rho f_rho alpha A;

sigma = .6;
 h_rho=0.80;
 f_rho=0.80;
 alpha=0.2;
 A=1;
Example with macros (continued)

model;
@#for c in countries
  @# for t in 1:T-1
      @{c}_c_{t}^{(-sigma)} = 
      @{c}_c_{t+1}(+1)^{(-sigma)}*(1+r(+1)) /
      (1+@{c}_rho+drho);
  @#if t == 1
      @{c}_s_{t} + @{c}_c_{t} = w;
  @#endif
  @#if (t > 1) && (t <= Ta)
      @{c}_s_{t} + @{c}_c_{t} = 
      w + (1+r)*@{c}_s_{t-1}(-1);
  @#endif
  @#if t > Ta
      @{c}_s_{t} + @{c}_c_{t} = 
      (1+r)*@{c}_s_{t-1}(-1);
  @#endif
@#endfor
@{c}_c_{T} = (1+r)*@{c}_s_{T-1}(-1);
@#endfor
\[ w = A(1 - \alpha) \cdot k(-1)^\alpha; \]
\[ r = A\alpha k(-1)^{\alpha-1}; \]
\[ (h_l(+1)+f_l(+1)) \cdot k = (h_s_1 @\#for t in 2:T-1
    +h_s_@\{t\}@\#endfor )\cdot h_l + (f_s_1 @\#for t in 2:T-1
    +f_s_@\{t\}@\#endfor )\cdot f_l; \]
end;
Example with macros (continued)

initval;
drho=0;
f_l=1;
h_l=1;

w=.5;
#@for c in countries
  #@for t in 1:T
    @{c}_c_{t} = 0.13;
    #@if t < T
      @{c}_s_{t} = 0.2;
    #@endif
  #@endfor
#@endfor
#@endfor

k=0.1;
r=1.0;
end;
Example with macros (continued)

steady;

d rval;
d rho=0.05;
w=1.0;
end;

steady;

simul(periods=20);

rplot h_c_1;
rplot r;