Effects of mis-specification of seasonal cointegrating ranks: An empirical study

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1 Byeongchan Seong’s research was supported by the Post-doctoral Fellowship Program of Korea Science & Engineering Foundation (KOSEF). The research of Sinsup Cho, S. Y. Hwang, and Sung K. Ahn was supported by the Korea Research Foundation Grant (KRF-2005-070-C00022) funded by the Korean Government (MOEHRD).
Co-integration (Engle & Granger, 1987)

An $m$-dimensional I($d$) process $y_t$ is co-integrated, if there exists a vector $\beta$ such that $\beta'y_t$ is an I($b$) process, $b < d$, denoted by CI($d$, $d-b$).

Typically, processes are CI(1, 0), i.e., $d=1$ & $b=0$.

The number of linearly independent vectors is called the co-integrating rank, denoted by $r$.

“Disappearance” of the non-stationarity, or unit root in $\beta'y_t$ is attributable to the common feature (Engle & Kozicki, 1993), more specifically called, common trend in some or all of the elements of $y_t$. 

Seasonal Co-integration
(Hylleberg, Engle, Granger & Yoo, 1990)

A seasonal process $y_t$ with period $s$ is seasonally co-integrated at frequency $f$, if there exists a vector $\beta$ such that $\beta'y_t$ does not have the seasonal unit root $e^{i\theta}$ corresponding to the frequency $f$, $\theta = 2\pi f$.

Since seasonal unit roots exist in conjugate pairs, there exists polynomial co-integrating vector $\beta_R + \beta_I L$ such that $(\beta_R + \beta_I L)'y_t$ does not have the seasonal unit root $e^{\pm i\theta}$.
The characteristics of (seasonal) co-integration are concentrated in the error correction terms through the reduced ranks of the coefficient matrices.

\[
\Phi^*(L)(1 - L)y_t = Cy_{t-1} + \varepsilon_t
\]

\[
\Phi^*(L)(1 - L^4)y_t = C_1 u_{t-1} + C_2 v_{t-1} + C_3 w_{t-1} + C_4 w_{t-2} + \varepsilon_t
\]

where

\[
u_{t-1} = (1 + L)(1 + L^2)y_{t-1},
\]

\[
v_{t-1} = (1 - L)(1 + L^2)y_{t-1}, \text{ and}
\]

\[
w_{t-1} = (1 - L^2)y_{t-1}
\]

Statistical inference of co-integration involves reduced rank estimation in the error correction representation of the vector autoregressive model.
Multivariate Regression Model

\[ z = C_1 x_1 + C_2 x_2 + C_3 x_3 + C_4 x_4 + \varepsilon \]

Estimation:

- Regression of $z$ on $x_1$, $x_2$, $x_3$, and $x_4$ simultaneously.
- Regression of $z$ on $x_j$ for each $j = 1, \ldots, 4$ if the $x_j$’s are uncorrelated.
- Partial regression of $z$ on $x_j$ adjusted for $x_k$, $k \neq j$ for each $j = 1, \ldots, 4$. 
Partial regression is especially useful if one of the $C_j$’s, say $C_1$ is of reduced rank. To estimate $C_1$ with the reduced rank structure imposed:

- Regress $z$ on $x_2, x_3, \text{ and } x_4$ and get the residual $r_z$;
- Regress $x_1$ on $x_2, x_3, \text{ and } x_4$ and get the residual $r_1$;
- Redused-rank regress $r_z$ on $r_1$, as in Anderson (1951).

In co-integration analysis

$$\Phi^* (L)(1 - L)y_t = Cy_{t-1} + \varepsilon_t$$

- Regress $(1 - L)y_t$ on lagged $(1 - L)y_t$ and get the residual $r_y$;
- Regress $y_{t-1}$ on lagged $(1 - L)y_t$ and get the residual $r_1$;
- Reduced-rank regress $r_y$ on $r_1$, as in Johansen (1988).
In seasonal co-integration analysis, more than one $C_j$’s in
$$z = C_1x_1 + C_2x_2 + C_3(\theta)x_3 + C_4(\theta)x_4 + \varepsilon$$
can be of reduced rank and some of the $C_j$’s are dependent on the common parameter vector.

If $C_1$ and $C_2$ are of reduced rank and $C_3$ and $C_4$ depend on $\theta$, then:

Since the adjustment for $x_2$, $x_3$, and $x_4$ is based on the full rank regression, partial reduced-rank regression of $z$ on $x_1$ is affected by over-specification of the rank of $C_2$;

Since the adjustment for $x_1$, $x_2$, and $x_3$ is based on the full rank regression, the dependence between $C_3$ and $C_4$ is ignored in partial (reduced-rank) regression of $z$ on $x_4$. 
Seasonal Co-integration

\[ \Phi^* (L)(1 - L^4)y_t = \alpha_{1R} \beta'_{1R} u_{t-1} + \alpha_{2R} \beta'_{2R} v_{t-1} + (\alpha_{3R} \beta'_{3I} + \alpha_{3I} \beta'_{3R}) w_{t-1} + (-\alpha_{3R} \beta'_{3R} + \alpha_{3I} \beta'_{3I}) w_{t-2} + \epsilon_t \]

Lee (1992), Johansen & Schaumburg (1999), and Cubadda (2001) use partial reduced rank regression exploiting asymptotic zero correlations:

- Lee assumes \( \alpha_{3I} = 0 \) and \( \beta_{3I} = 0 \);
- J&S uses the “switching” algorithm to estimate \( \alpha_{3R} \), \( \beta_{3R} \), \( \alpha_{3I} \), and \( \beta_{3I} \);
- Cubadda, in essence, estimates \( \alpha_{3R} \), \( \beta_{3R} \), \( \alpha_{3I} \), and \( \beta_{3I} \) based on partial regression of \( (1 - L^4)y_t \) on \( w_{t-1} \).

These create over-specification problems.
Ahn & Reinsel (1994) and Ahn, Cho & Seong (2004) use an iterative scheme that incorporates
  • the co-integrating ranks at all the seasonal frequencies simultaneously and
  • the dependency among the coefficient matrices.

But this requires the correct specification of the seasonal co-integrating ranks and is subject to over- and under-specification. (Furthermore, it can be computationally challenging.)
Simulation Study

DGP (Ahn & Reinsel, 1994):

\[(1 - L^4)y_t = \alpha_1 \beta_1 u_{t-1} + \alpha_2 \beta_2 v_{t-1} \]

\[+ (\alpha_3 \beta_4 + \alpha_4 \beta_3) w_{t-1} \]

\[+ (-\alpha_3 \beta_3 + \alpha_4 \beta_4) w_{t-2} + \epsilon_t \]

where

\[\alpha_1 = (a_{11}, a_{21})' = (0.6, 0.6)',\]

\[\alpha_2 = (a_{12}, a_{22})' = (-0.4, 0.6)',\]

\[\alpha_3 = (a_{13}, a_{23})' = (0.6, -0.6)',\]

\[\alpha_4 = (a_{14}, a_{24})' = (0.4, -0.8)',\]

\[\beta_1 = (1, b_1)' = (1, -0.7)', \quad \beta_2 = (1, b_2)' = (1, 0.3)',\]

\[\beta_3 = (1, b_3)' = (1, 0.7)', \quad \beta_4 = (0, b_4)' = (0, -0.2)',\]
\[ \text{Cov}(\varepsilon_t) = \Omega = \begin{pmatrix} 1 & \rho \sigma \\ \rho \sigma & \sigma^2 \end{pmatrix} \]

for \( \rho = -0.5, 0, 0.5 \) and \( \sigma^2 = 0.5, 1, 2 \).

Series length: 100

Replications: 1000

Nominal size: 0.05


For \( H_0 : r_f = 0 \) vs \( H_1 : r_f > 0 \) for 
\( f = 0, 1/2, 1/4 \), \( H_0 \) is rejected almost all the cases regardless of under or over-specification.

For \( H_0 : r_f = 1 \) vs \( H_1 : r_f > 1 \), the results are summarized below.
Table 1. Comparison of the rejection rates of 5% level tests for hypotheses in (6) for the frequency $f = 1/2$.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$\sigma^2$</th>
<th>C.I. ranks $(r_0, r_{1/4}, r_{1/2})$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>(0,0,1)</td>
</tr>
<tr>
<td>-0.5</td>
<td>0.5</td>
<td>0.084</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.084</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.081</td>
</tr>
<tr>
<td>0</td>
<td>0.5</td>
<td>0.086</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.083</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.088</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.082</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.075</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.079</td>
</tr>
</tbody>
</table>

- Significantly larger empirical sizes with under-specification for $f=0$ & $1/2$.
- Significantly smaller empirical sizes with under-specification for only one of $f=0$ & $1/2$. 
<table>
<thead>
<tr>
<th>CI ranks ( (r_0, r_{1/4}, r_{1/2}) )</th>
<th>( a_{12} = -0.4 ) Mean</th>
<th>( a_{12} = -0.4 ) MSE</th>
<th>( a_{22} = 0.6 ) Mean</th>
<th>( a_{22} = 0.6 ) MSE</th>
<th>( b_2 = 0.3 ) Mean</th>
<th>( b_2 = 0.3 ) MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0,1)</td>
<td>-0.3060</td>
<td>0.0182</td>
<td>0.7821</td>
<td>0.0624</td>
<td>0.2938</td>
<td>0.0002</td>
</tr>
<tr>
<td>(0,1,1)</td>
<td>-0.2391</td>
<td>0.0541</td>
<td>0.6164</td>
<td>0.0566</td>
<td>0.2989</td>
<td>0.0004</td>
</tr>
<tr>
<td>(0,2,1)</td>
<td>-0.2521</td>
<td>0.0462</td>
<td>0.6031</td>
<td>0.0534</td>
<td>0.2977</td>
<td>0.0009</td>
</tr>
<tr>
<td>(1,0,1)</td>
<td>-0.3186</td>
<td>0.0148</td>
<td>0.6789</td>
<td>0.0196</td>
<td>0.2931</td>
<td>0.0002</td>
</tr>
<tr>
<td>(1,1,1)</td>
<td>-0.3621</td>
<td>0.0107</td>
<td>0.5145</td>
<td>0.0443</td>
<td>0.3014</td>
<td>0.0000</td>
</tr>
<tr>
<td>(1,2,1)</td>
<td>-0.3603</td>
<td>0.0102</td>
<td>0.5160</td>
<td>0.0430</td>
<td>0.3014</td>
<td>0.0001</td>
</tr>
<tr>
<td>(2,0,1)</td>
<td>-0.3099</td>
<td>0.0148</td>
<td>0.6701</td>
<td>0.0181</td>
<td>0.2938</td>
<td>0.0001</td>
</tr>
<tr>
<td>(2,1,1)</td>
<td>-0.3600</td>
<td>0.0108</td>
<td>0.5038</td>
<td>0.0463</td>
<td>0.3014</td>
<td>0.0001</td>
</tr>
<tr>
<td>(2,2,1)</td>
<td>-0.3581</td>
<td>0.0104</td>
<td>0.5055</td>
<td>0.0449</td>
<td>0.3014</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

- Serious biases occur with under-specification for the stationary parameters.
- Biases are not serious with under-specification for the long-run parameter.
Table 2. Comparison of the rejection rates of 5% level tests for hypotheses in (6) for the frequency \( f = 0 \).

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( \sigma^2 )</th>
<th>CI ranks ((r_0, r_{1/4}, r_{1/2}))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( (1,0,0) )</td>
<td>( (1,0,1) )</td>
</tr>
<tr>
<td>-0.5</td>
<td>0.5</td>
<td>0.249</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.262</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.273</td>
</tr>
<tr>
<td>0</td>
<td>0.5</td>
<td>0.318</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.340</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.376</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.386</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.445</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.494</td>
</tr>
</tbody>
</table>

- Significantly larger empirical sizes with under-specification.
- Significantly larger empirical sizes with under-specification for \( f = 1/4 \) and over-specification for \( f = 1/2 \).
- Significantly smaller empirical sizes with over-specification for \( f = 1/4 \) and under-specification for \( f = 1/2 \).
Table 3. Comparison of the rejection rates of 5% level tests for hypotheses in (6) for the frequency $f = 1/4$.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$\sigma^2$</th>
<th>C.I. ranks ($r_0, r_{1/4}, r_{1/2}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>(0,1,0) (0,1,1) (0,1,2) (1,1,0) (1,1,1) (1,1,2) (2,1,0) (2,1,1) (2,1,2)</td>
</tr>
<tr>
<td>-0.5</td>
<td>0.5</td>
<td>0.093 0.058 0.066 0.019 0.020 0.017 0.018 0.019 0.017</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.132 0.100 0.105 0.020 0.018 0.019 0.020 0.017 0.018</td>
</tr>
<tr>
<td></td>
<td>2.000</td>
<td>0.167 0.138 0.143 0.019 0.016 0.016 0.020 0.016 0.016</td>
</tr>
<tr>
<td>0</td>
<td>0.5</td>
<td>0.083 0.048 0.055 0.024 0.025 0.024 0.024 0.025 0.024</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.112 0.091 0.094 0.027 0.029 0.030 0.027 0.029 0.031</td>
</tr>
<tr>
<td></td>
<td>2.000</td>
<td>0.137 0.113 0.116 0.030 0.024 0.026 0.030 0.025 0.026</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.139 0.094 0.095 0.073 0.075 0.079 0.074 0.076 0.080</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.165 0.128 0.134 0.075 0.084 0.086 0.076 0.084 0.083</td>
</tr>
<tr>
<td></td>
<td>2.000</td>
<td>0.199 0.164 0.167 0.078 0.077 0.077 0.079 0.079 0.078</td>
</tr>
</tbody>
</table>

- Significantly larger empirical sizes with under-specification for both $f=0$ and $1/4$ and for only $f=0$.
- No significant difference with under-specification for only $f=1/2$.
- Significantly larger empirical sizes with under-specification for $f=0$ and over-specification for $f=1/2$.
- No significant difference with over-specification for $f=0$ and under-specification for $f=1/2$. 
Summary

• Over specification of co-integrating ranks is acceptable, and so is the partial regression based approach.

• May need to check the validity of the critical values.

• Further simulation study is needed.

• Theoretical investigation in needed.