Effects of mis-specification of seasonal cointegrating ranks: 
An empirical study

Byeongchan Seong
Sinsup Cho
S. Y. Hwang
Sung K. Ahn
Effects of mis-specification of seasonal cointegrating ranks: An empirical study

Byeongchan Seong\textsuperscript{a}, Sinsup Cho\textsuperscript{b}, S. Y. Hwang\textsuperscript{c}, and Sung K. Ahn\textsuperscript{d}

\textsuperscript{a} Department of Mathematics, Pohang University of Science and Technology, Pohang, Kyeongbuk, 790-784, KOREA (e-mail: bcseong@postech.ac.kr)
\textsuperscript{b} Department of Statistics, Seoul National University, Seoul, 151-747, KOREA (e-mail: sinsup@snu.ac.kr)
\textsuperscript{c} Department of Statistics, Sookmyung Women’s University, Seoul, 140-742, KOREA (e-mail: shwang@sookmyung.ac.kr)
\textsuperscript{d} Department of Management and Operations, Washington State University, Pullman, WA 99164-4736, USA (e-mail: ahn@wsu.edu)

Abstract

We investigate the effects of mis-specification of cointegrating ranks at other frequencies on inference of seasonal cointegration at the frequency of interest such as test for cointegrating rank and estimation of cointegrating vector. Earlier studies focused mostly on a single frequency corresponding to a seasonal root at a time, ignoring possible cointegration at the remaining frequencies. We investigate the effects of the mis-specification, especially with finite samples adopting Gaussian reduced rank estimation by Ahn and Reinsel (1994) that considered cointegration at all frequencies of seasonal unit roots simultaneously. It is observed that the identification of the seasonal cointegrating

\textsuperscript{1} Byeongchan Seong’s research was supported by the Post-doctoral Fellowship Program of Korea Science & Engineering Foundation (KOSEF). The research of Sinsup Cho, S. Y. Hwang, and Sung K. Ahn was supported by the Korea Research Foundation Grant (KRF-2005-070-C00022) funded by the Korean Government (MOEHRD).
rank at the frequency of interest is robust to over-specification of the cointegrating ranks at other frequencies, especially, while very sensitive to under-specification.

**Keywords:** Gaussian reduced rank estimation, seasonal cointegration

1. Introduction

Since Hylleberg *et al.* (1990), several approaches to analysis of seasonal cointegration have been developed. Because these approaches are based on vector autoregressive models we summarize them in the context of multivariate regression, and address issues associated with them. To this end, we consider the following multivariate regression model

\[ z = C_1 x_1 + C_2 x_2 + C_3 x_3 + C_4 x_4 + \varepsilon, \]  

where \( z \) is an \( m \)-dimensional random vector, \( x_j \) is an \( m_j \)-dimensional random vector and \( C_j \) is an \( m \times m_j \) matrix for \( j = 1, \ldots, 4 \), and \( \varepsilon \) is an \( m \)-dimensional error vector.

It is well known that, when the \( C_j \)'s are of full rank and unrestricted, each of the coefficient matrices may be estimated one at a time by regressing \( z \) on the \( x_j \) adjusting for \( x_k \) for \( k \neq j \), that is, by regressing the residuals from regression of \( z \) on the \( x_k \)'s on the residuals from regression of \( x_j \) on the \( x_k \)'s for \( k \neq j \), known as partial regression.

Anderson (1951) employed this idea of partial regression and proposed a way to estimate when one of the coefficient matrices, for example, \( C_1 \) is of reduced rank. This involves reduced-rank regression of \( z \) on \( x_1 \) adjusted for the \( x_k \)'s for \( k \neq 1 \), called
partial reduced-rank regression. In (non-seasonal) cointegration analysis, Johansen (1988) used this partial reduced rank regression approach because the error correction term in the model is of reduced rank, while the other coefficient matrices are of full rank.

In the context of seasonal cointegration, there may be more than one coefficient matrices that are of reduced rank in (1). In such cases, estimation of a reduced-rank matrix, one at a time using partial reduced rank regression, amounts to estimation of the reduced-rank matrix treating the other reduced-rank matrices as of full rank and unrestricted, that is, with over-specification of the (seasonal cointegrating) ranks of these matrices. This approach has been adopted in Lee (1992), Johansen and Schaumburg (1999), and Cubadda (2001) because it is simple to implement and the regressors, $x_ε$’s, are asymptotically uncorrelated.

If the $x_j$’s are uncorrelated, the matrix $C_j$ can be estimated by (reduce-rank) regression of $z$ on $x_j$ without any adjustment for the other regressors. In this approach, an under-specification of the ranks of the other coefficients occurs in finite samples, when the $x_j$’s are uncorrelated only asymptotically.

Furthermore, in the seasonal cointegration context, not only the $C_j$’s are of reduced rank but also some of them are not functionally independent. When, for example, $C_3$ and $C_4$ in (1) are functions of a common parameter vector $θ$, we re-express the model as

$$z = C_1x_1 + C_2x_2 + C_3(θ)x_3 + C_4(θ)x_4 + ε$$

(2)
in order to emphasize the dependence on the $θ$. In this case $C_3$ and $C_4$ need simultaneous estimation through an estimation of the $θ$. However, Cubadda (2001)
estimates \( C_3 \) and \( C_4 \) separately using aforementioned partial reduce-rank regression.

As in the case of seasonal cointegration, if it is known that the coefficient matrices are of reduced rank and there are functional dependencies among them, then a sensible approach is to incorporate these and estimate the parameters simultaneously. Ahn and Reinsel (1994) and Ahn et al. (2004) used an iterative scheme that incorporates the cointegrating ranks at all the seasonal frequencies simultaneously and the dependency among the coefficient matrices. However, this requires the correct specification of the cointegrating ranks, and thus subject to over- and under-specification of the ranks.

Because the aforementioned approaches of Lee (1992), Johansen and Schaumburg (1999), and Cubadda (2001) are subject to over-specification and Ahn and Reinsel (1994) and Ahn et al. (2004) are subject to both over- and under-specification, we investigate the effects of mis-specification through the Monte Carlo simulation.

2. Seasonally Cointegrated Time Series and Its Estimations

Let \( y_t \) be an \( m \)-dimensional time series with non-stationary seasonal behavior and period \( s \) such that

\[
\Phi(L)y_t = (I_m - \sum_{j=1}^{p} \Phi_j L^j)y_t = \Pi D_t + \varepsilon_t, \tag{3}
\]

where \( \varepsilon_t \) are i.i.d. \( N_m(0, \Omega) \), and \( D_t \) is a deterministic term that may contain a constant, a linear term, or seasonal dummies. We assume that the initial values \( y_0, \ldots, y_{-p+1} \) are fixed and that the roots of the determinant \( |\Phi(z)| = 0 \) are on or outside the unit circle.

For simplicity of presentation, we consider the series \( y_t \) observed quarterly, that is,
If the series are cointegrated at frequencies 0, 1/2, and 1/4 (0, \(\pi\), and \(\pi/2\), respectively), model (3) may be rewritten in the following error correction model (ECM) as in Ahn and Reinsel (1994) and Ahn et al. (2004).

\[
\Phi^*(L)(1-L^4)y_t = \prod \Delta_t + \alpha_{1R} \beta'_{1R}(1+L)(1+L^2)y_{t-1} + \alpha_{2R} \beta'_{2R}(1-L)(1+L^2)y_{t-1} \\
+ (\alpha_{3R} \beta'_{3R} + \alpha_{3I} \beta'_{3I})(1-L^2)y_{t-1} + (\alpha_{3R} \beta'_{3R} + \alpha_{3I} \beta'_{3I})(1-L^2)y_{t-2} + \epsilon_t,
\]

where \(\alpha_{1R} \beta'_{1R} = -\Phi(1)/4\), \(\alpha_{2R} \beta'_{2R} = \Phi(-1)/4\), \((\alpha_{3R} + i\alpha_{3I})(\beta_{3R} + i\beta_{3I}) = -\Phi(i)/2\), \(\alpha_{3R}\), \(\beta_{3R}\), and \(\beta_{3I}\) are \(m \times r\) real-valued matrices with a rank equal to \(r\), \(\beta_{jR} = [I_{r_j}, \beta'_{0jR}]\), and \(\beta_{jI} = [O_{r_j}, \beta'_{0jI}]\); \(I_{r_j}\) and \(O_{r_j}\) are an \(r_j \times r_j\) identity matrix and an \(r_j \times r_j\) zero matrix, respectively. We note that the subscript \(j = 1, 2\) and 3 corresponds to the frequencies 0, 1/2 and 1/4, respectively. We also note that the model in (4) takes the form of the model in (2) with \(\Phi^*(L) = I\), \((1-L^4)y_t = z\),

\[(1+L)(1+L^2)y_{t-1} = x_1, \quad (1-L)(1+L^2)y_{t-1} = x_2, \quad (1-L^2)y_{t-1} = x_3, \quad (1-L^2)y_{t-2} = x_4, \]

\(\alpha_{1R} \beta'_{1R} = C_1\), \(\alpha_{2R} \beta'_{2R} = C_2\), \(\alpha_{3R} \beta'_{3R} + \alpha_{3I} \beta'_{3I} = C_3\), and \(-\alpha_{3R} \beta'_{3R} + \alpha_{3I} \beta'_{3I} = C_4\), and also note that \(C_3\) and \(C_4\) depend on the common parameters \(\alpha_{3R}\), \(\alpha_{3I}\), \(\beta_{3R}\), and \(\beta_{3I}\).

The Gaussian reduced rank estimation of Ahn and Reinsel (1994) uses the iterative Newton-Raphson method that can be computationally complicated because of the non-linear nature of the parameters \(\alpha_j\) ’s and \(\beta_j\) ’s. Its advantage is to simultaneously incorporate the reduced-rank structures attributable to cointegration at all different frequencies in model (4). Therefore, the efficiency of inference such as the test for
cointegrating ranks and estimation of cointegrating vectors may be improved for finite samples. However, it requires the correct specification of the cointegrating ranks at all seasonal frequencies, and it is not easy to correctly specify these ranks simultaneously. We may pre-specify the cointegrating rank at each of the frequencies ignoring the CI ranks at the other frequencies, for example, by partial reduced ranks regression as in Cubadda (2001). This pre-specification may cause over- or under-specification of some of the cointegrating ranks. Although we may be able to correctly specify the cointegrating ranks by applying the likelihood ratio test iteratively, it is interesting to investigate the effects of mis-specification of the cointegrating ranks.

The approaches of Lee (1992), Johansen and Schaumburg (1999), and Cubadda (2001) are different from the Gaussian reduced rank estimation of Ahn and Reinsel (1994). As the terms \((1 + L)(1 + L^2)y_{t-1}, (1 - L)(1 + L^2)y_{t-1},\) and \((1 - L^2)y_{t-1}\) on the right side of model (4) are asymptotically uncorrelated, they focused on a single frequency at a time by adjusting for the terms at the other frequencies. Because the adjustment is based on full rank regression, the reduced-rank structures at the other frequencies are treated as of full rank. This amounts to over-specification of the cointegrating ranks. Alternatively, one may consider reduced rank regression without the adjustment because of the asymptotic uncorrelatedness among those terms. Then, this amounts to under-specification of the cointegrating ranks. These approaches may be simpler than that of Ahn and Reinsel (1994) but may accompany with a loss of efficiency. Therefore, it may be interesting to investigate the efficiency in inference of cointegration when over- or under-specification occurs.
3. Monte Carlo experiment

In this section, we conduct a Monte Carlo experiment to investigate the effects of mis-specification of cointegrating ranks at other frequencies on the test for the cointegrating rank and estimation of cointegrating vectors at the frequency of interest by adopting the Gaussian reduced rank estimation.

For a Monte Carlo experiment, the data generating process considered is the bivariate quarterly VAR(4) process \( y_t = \sum_{j=1}^{4} \Phi_j y_{t-j} + \varepsilon_t \), which has the following error correction representation:

\[
(1-L^1)y_t = \alpha_1^t_1 y_{t-1} + \alpha_2^t_2 v_{t-1} + (\alpha_3^t_3 + \alpha_4^t_4)w_{t-1} + (-\alpha_5^t_5 + \alpha_6^t_6)w_{t-2} + \varepsilon_t,
\]

where \( \alpha_1 = (a_1, a_2) = (0.6, 0.6)' \), \( \alpha_2 = (a_1, a_2) = (-0.4, 0.6)' \),
\( \alpha_3 = (a_1, a_2) = (0.6, -0.6)' \), \( \alpha_4 = (a_1, a_2) = (0.4, -0.8)' \), \( \beta_1 = (1, b_1)' = (1, -0.7)' \), \( \beta_2 = (1, b_2)' = (1, 0.3)' \), \( \beta_3 = (1, b_3)' = (1, 0.7)' \), \( \beta_4 = (0, b_4)' = (0, -0.2)' \),
\( u_{t-1} = (1 + L)(1 + L^2)y_{t-1} \), \( v_{t-1} = (1 - L)(1 + L^2)y_{t-1} \), and \( w_{t-1} = (1 - L^2)y_{t-1} \). The covariance matrix \( \Omega \) of \( \varepsilon_t \) is taken to be

\[
\Omega = \begin{pmatrix} 1 & \rho \sigma \\ \rho \sigma & \sigma^2 \end{pmatrix}
\]

for \( \rho = -0.5, 0, 0.5 \) and \( \sigma^2 = 0.5, 1, 2 \). We note that the roots of the characteristic equation \( \det \{ \Phi(L) \} = 0 \) are \( \pm 1, \pm i, 0.9715 \pm 0.7328i \) and \( -1.3508 \pm 0.3406i \), and \( y_t \) is seasonally cointegrated with cointegrating rank of one at each of the frequencies 0, 1/2 and 1/4.
For the effects on test for cointegrating rank at given a frequency, we are interested in the following hypothesis:

\[ H_0 : r_f \leq 1 \quad \text{vs} \quad H_1 : r_f > 1 \quad \text{for} \quad f = 0, 1/2, 1/4. \quad (6) \]

In each test at given a frequency \( f \), the cointegrating ranks at other frequencies are set to be under-specified as 0, exact-specified as 1, or over-specified as 2. (We omit a report for the powers of the cointegrating rank test: \( H_0 : r_f = 0 \) because the empirical powers were almost all 100% irrespective of mis-specification of cointegrating ranks at the other frequencies.) For testing the hypotheses, the likelihood ratio test statistic is used:

\[
LR = 2 \ln \left( \frac{\max_{H_1} L}{\max_{H_0} L} \right) = -T \ln \left( \frac{\max_{H_1} |\hat{\Omega}|}{\max_{H_0} |\hat{\Omega}|} \right),
\]

where \( L \) is the likelihood function based on the normality assumption and \( \hat{\Omega} \) is the covariance matrix of the residuals resulting from the Gaussian reduced rank estimation.

Samples of series with length \( T = 100 \) are generated with 1,000 replications. Initial values are set to zero, and we discard the first 50 observations in order to alleviate dependence on the initial values. The estimated model is a VAR(5) with unrestricted seasonal deterministic terms. All the tests are based on the 5% asymptotic critical values that are obtained from Johansen and Schaumburg (1999) and Lee and Siklos (1995).

The results of the simulations are summarized in Tables 1, 2, and 3 and Figures 1, 2, and 3. The tables show the rejection rates of the tests for the hypothesis in (6) in all cases for \( f = 1/2, 0, \) and \( 1/4, \) respectively, which are displayed in the figures for some selected cases. In these tables, the columns under the triplet \((1,1,1)\) are the empirical sizes, when the cointegrating ranks at the other frequencies are correctly specified.
For the tests at $f = 1/2$ summarized in Table 1, the empirical sizes are not significantly different from the nominal level of 0.05, in general. When the cointegrating rank at $f = 0$ or $1/4$ is over-specified, the empirical sizes that are displayed under the triplets (1,2,1), (2,1,1), and (2,2,1), are larger than those of the tests with the correct rank specification at these two frequencies, but not significantly different. When the cointegrating ranks at $f = 0$ and $1/4$ are under-specified, the empirical sizes that are displayed under the triplet (0,0,1), are significantly larger than those of the tests with the correct rank specification at these two frequencies. When the cointegrating rank at only one of $f = 0$ and $1/4$ is under-specified, the empirical sizes that are displayed under the triplets (0,1,1) and (1,0,1), are significantly smaller than those of the tests with the correct rank specification at these two frequencies except for the case with $\rho = -0.5$ and $\sigma^2 = 2$. When the under-specification occurs at one of the frequencies and over-specification at the other, the empirical sizes are smaller and those with under-specification at $f = 0$ and over-specification at $f = 1/4$ are much smaller in general.

For the tests at $f = 0$ summarized in Table 2, the empirical sizes are significantly smaller than the nominal size, and get smaller as the correlation $\rho$ change from negative to positive. These size biases for the tests at $f = 0$ may be attributable to inappropriate uses of the critical values that are obtained based on large samples, while the sample size of 100 in this study is small. The effect of over-specification is similar to that of over-specification for the tests at $f = 1/2$. However, significant upward size biases (compared with the empirical sizes with the correct rank specification at all the frequencies) occurs when the under-specification occurs at $f = 1/4$. When under-specification occurs at $f = 1/4$. When under-specification occurs at
$f = 1/2,$ there are no significant differences among the empirical sizes with $\rho = -0.5,$ while the empirical sizes are significantly larger with $\rho = 0$ and 0.5.

For the tests at $f = 1/4$ summarized in Table 3, the empirical sizes are significantly smaller for $\rho = -0.5, 0$ and significantly larger for $\rho = 0.5$ than the nominal level when the ranks are correctly specified at the other frequencies. These size biases can also be attributable to inappropriate uses of the critical values that are obtained based on large samples, while the sample size of 100 in this study is small. The different directions of the size biases need further investigation. The effect of over-specification is similar to that of over-specification for the tests at $f = 1/2$. Significant upward size biases (compared with the empirical sizes with the correct rank specification at all the frequencies) occur when the under-specification occurs at $f = 0$. When under-specification occurs at $f = 1/2$ only, there are no significant differences among the empirical sizes.

The interesting observation is that for $f = 0$ and $1/4$ the results are symmetric in that the empirical sizes of the tests for one of the frequencies are significantly larger when the cointegrating rank of the other frequency is under-specified. However, the commonly-accepted fact that the frequencies $f = 0$ and $1/2$ are symmetric does not appear to hold in our simulation for the finite sample.

Based on the limited simulation study presented here, it is recommended that in pre-specifying the cointegrating ranks, over-specification is generally preferred to under-specification and that under-specification is avoided, especially, at the zero frequency.
References


Table 1. Comparison of the rejection rates of 5% level tests for hypotheses in (6) for the frequency $f = 1/2$.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$\sigma^2$</th>
<th>C.I. ranks $(r_0, r_{i/4}, r_{i/2})$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(0,0,1)</td>
<td>(0,1,1)</td>
</tr>
<tr>
<td>-0.5</td>
<td>0.5 0.084  0.024 0.029 0.037 0.050 0.055 0.035 0.050 0.055</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1  0.084  0.037 0.043 0.039 0.058 0.059 0.040 0.059 0.059</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2  0.081  0.062 0.070 0.035 0.060 0.061 0.034 0.059 0.061</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.5 0.086  0.026 0.029 0.035 0.056 0.059 0.040 0.057 0.062</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1  0.083  0.035 0.038 0.040 0.060 0.065 0.042 0.060 0.065</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2  0.088  0.053 0.056 0.041 0.064 0.066 0.040 0.063 0.066</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.5 0.082  0.020 0.025 0.045 0.056 0.058 0.051 0.055 0.057</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1  0.075  0.025 0.033 0.040 0.060 0.065 0.044 0.060 0.063</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2  0.079  0.043 0.047 0.043 0.064 0.063 0.046 0.063 0.063</td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Comparison of the rejection rates of 5% level tests for hypotheses in (6) for the frequency $f = 0$.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$\sigma^2$</th>
<th>CI ranks $(r_0, r_{i/4}, r_{i/2})$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1,0,0)</td>
<td>(1,0,1)</td>
</tr>
<tr>
<td>-0.5</td>
<td>0.5 0.249  0.233 0.236 0.028 0.025 0.025 0.025 0.023 0.025</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1  0.262  0.256 0.262 0.030 0.022 0.024 0.031 0.024 0.024</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2  0.273  0.276 0.280 0.035 0.025 0.025 0.038 0.026 0.026</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.5 0.318  0.323 0.324 0.028 0.014 0.016 0.030 0.013 0.016</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1  0.340  0.345 0.351 0.039 0.018 0.019 0.041 0.019 0.019</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2  0.376  0.374 0.373 0.047 0.020 0.020 0.047 0.022 0.023</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.5 0.386  0.401 0.407 0.040 0.013 0.013 0.039 0.013 0.015</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1  0.445  0.464 0.467 0.053 0.014 0.014 0.052 0.014 0.014</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2  0.494  0.516 0.526 0.061 0.014 0.014 0.059 0.014 0.014</td>
<td></td>
</tr>
</tbody>
</table>
Table 3. Comparison of the rejection rates of 5% level tests for hypotheses in (6) for the frequency \( f = 1/4 \).

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( \sigma^2 )</th>
<th>C.I. ranks ( (r_0, r_{1/4}, r_{1/2}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.093</td>
<td>0.058</td>
</tr>
<tr>
<td>1</td>
<td>0.132</td>
<td>0.100</td>
</tr>
<tr>
<td>2</td>
<td>0.167</td>
<td>0.138</td>
</tr>
<tr>
<td>0</td>
<td>0.5</td>
<td>0.083</td>
</tr>
<tr>
<td>1</td>
<td>0.112</td>
<td>0.091</td>
</tr>
<tr>
<td>2</td>
<td>0.137</td>
<td>0.113</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.139</td>
</tr>
<tr>
<td>1</td>
<td>0.165</td>
<td>0.128</td>
</tr>
<tr>
<td>2</td>
<td>0.199</td>
<td>0.164</td>
</tr>
</tbody>
</table>