An alternative framework for univariate and multivariate seasonal adjustment

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CREST - INSEE

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Statisticians do not agree on a definition of the seasonal component of a time series: Any rigorous statistical definition can only be based on a model of the series, and there is no agreement on a common model between the different users or the statisticians.

The usual model assumes that the series under study can be decomposed into the sum of uncorrelated components, one of them generating the seasonal pattern.
The current practices fail to meet the needs of most users: the usual seasonal adjustment statistical procedures are univariate but practitioners look at a comprehensive set of statistics (production, household purchases, employment, credit, wages and prices...), (Porter (1975), Geweke (1979), Plosser (1979) and Wallis (1979)).

In absence of consistent treatment between various variables of interest, medium and long run comovements may be distorted (Sims (1974), Wallis (1974))
The impact of outliers: Outliers are identified with respect to the selected model. This identification is contingent upon the sample to hand and not necessarily time consistent. An observation detected as an outlier for a given sample size may appear as a regular observation in a larger sample.

The current data point is looked at with most attention, but it is unfortunately the less reliable for at least two reasons: it is subject to preliminary-data error and it is obtained by applying either a non-optimal one sided filter, or a two sided filter to forecasts (and backcasts) of the raw data.

Challenge: in case of outliers close to the end of the sample that simultaneously affects several variables...
Introduction of a simple linear method...

...that allows for a multivariate treatment...

...that allows for the use of robust estimation procedure (outliers)... 

...that gives a rationale for the computation of asymmetric filter at the end of the sample (DGP dependent)... 

....properties of which cannot be analyzed in the usual stationary framework (the DGP dependent filters are applied on non-stationary time series)... 

...that does not assume the absence of correlation between the seasonal component and the other ones.
Outline

1. Seasonal random walks and demodulated Seasonal random walks
2. Persistence at seasonal frequency
   - Integral operator algebra
   - Persistent components
3. Seasonal adjustment by subtraction of the seasonally persistent components
   - Introductory example: the Airline model
   - Univariate processes
4. Practical implementation in a SARIMA framework
   - Unit root testing
   - Demodulation and long term persistent component
5. Illustration
   - Monthly time series
   - Quarterly time series
Seasonal random walks and demodulated Seasonal random walks

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- Integral operator algebra
- Persistent components

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Illustration
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- Quarterly time series
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   - Persistent components
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   - Univariate processes
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   - Unit root testing
   - Demodulation and long term persistent component
5. Illustration
   - Monthly time series
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   - Univariate processes
4. Practical implementation in a SARIMA framework
   - Unit root testing
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   - Monthly time series
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Seasonal random walk (complex process): unit root at frequency \( \omega \in ]0, \pi[ \)

\[ y_t = e^{i\omega} y_{t-1} + \epsilon_t \]

Real seasonal random walk: two unit roots \( e^{i\omega}, e^{-i\omega} \)

\[ y_t = 2 \cos(\omega) y_{t-1} - y_{t-2} + \epsilon_t \]

**Example**

\[ \omega = \frac{\pi}{2} \quad y_t = -y_{t-2} + \epsilon_t \]
Need for evolving seasonal patterns (2)
Seasonal random walks and demodulated Seasonal random walks
Persistence at seasonal frequency
Seasonal adjustment by subtraction of the seasonally persistent components
Practical implementation in a SARIMA framework
Illustration

Need for evolving seasonal patterns (2)
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Demodulation: \( \{ e^{i\omega t} x_t \} \)
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Seasonal adjustment by subtraction of the seasonally persistent components
Practical implementation in a SARIMA framework
Illustration

Demodulation: \( \{ e^{i \omega t} x_t \} \)
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Practical implementation in a SARIMA framework
Illustration

Demodulation: \( \{e^{i\omega t} x_t\} \)

demodulated seasonal random walk

demodulated seasonal random walk

demodulated seasonal random walk
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2. **Persistence at seasonal frequency**
   - Integral operator algebra
   - Persistent components
3. Seasonal adjustment by subtraction of the seasonally persistent components
   - Introductory example: the Airline model
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An alternative framework for univariate and multivariate seasonal adjustment - Stéphane Gregoir
Processus integrated at seasonal frequencies

First-difference operators at various frequencies are denoted as follows:

\[
\Delta_\omega(B) = \begin{cases} 
\delta_\omega(B) = 1 - e^{-i\omega}B, & \omega \in \{0, \pi\} \\
1 - 2\cos\omega B + B^2 = \begin{cases} 
(1 - e^{-i\omega}B)(1 - e^{i\omega}B) & , \quad \omega \in ]0, \pi[
\delta_\omega(B)\delta_{-\omega}(B)
\end{cases}
\end{cases}
\]

where \( B \) is the backshift operator.

**Definition**

A purely non deterministic real random process \( \{x_t\} \) is said to be integrated of order \( h, \ h \) integer, at frequency \( \omega, \ \omega \in [0, \pi] \), or \( I_\omega(h) \), if \( \Delta_\omega(B)^hx_t \) is a covariance stationary process such that the series \( c(u) \) associated with its Wold representation satisfies \( c(e^{i\omega}) \neq 0 \) (consequently \( c(e^{-i\omega}) \neq 0 \) when \( \omega \in ]0, \pi[ \)).
Integral operators

Definition

The integral operator $S_\omega$ associates to any sequence $y_t = (y_t, t = \ldots, -1, 0, 1, \ldots)$ of $\mathbb{R}^\mathbb{Z}$ a sequence $S_\omega y$ defined by:

$$S_\omega y_t = \begin{cases} \sum_{\tau=1}^{t} y_\tau e^{-i\omega(t-\tau)} & t \geq 1 \\ 0 & t = 0 \\ -\sum_{\tau=t+1}^{0} y_\tau e^{-i\omega(t+1-\tau)} & t < 0. \end{cases}$$
Basic properties of Integral operators

- \( \forall \omega \in [-\pi, \pi], \quad \delta_\omega S_\omega = I \) \hfill (1)

- \( \forall y_t \in \mathbb{R} \)

  \[ \omega \in [0, \pi], \quad S_\omega \delta_\omega y_t = y_t - y_0 e^{-i\omega t} \] \hfill (2)

- \( \forall (\omega_1, \omega_2) \in [0, \pi]^2, \omega_1 \neq \omega_2 \)

  \[ S_{\omega_1} S_{\omega_2} = \frac{e^{i\omega_1}}{e^{i\omega_1} - e^{i\omega_2}} S_{\omega_2} + \frac{e^{i\omega_2}}{e^{i\omega_2} - e^{i\omega_1}} S_{\omega_1} \] \hfill (3)
Decomposition into a persistent and a transitory component for processes integrated at frequency 0

- $I(1)$ processes: measure of shock persistence (Campbell and Mankiw (1987), Cochrane (1988))

- $I(1)$ processes: decompositions into the sum of a permanent and a transitory components (Quah (1994))

- Beveridge-Nelson decomposition: the permanent component is a pure random walk
Decomposition into a persistent and a transitory component for processes integrated at seasonal frequencies

Let \( \{x_t\} \) be a purely nondeterministic (complex) time series integrated at only one frequency \( \omega \in [0, \pi] \), the Wold representation of the covariance stationary representation of this process is given by:

\[
\delta_\omega (B) x_t = c(B) \varepsilon_t
\]

where \( \{\varepsilon_t\} \) is a weak white noise of variance \( \sigma_{\varepsilon}^2 \).

**Condition A**

\[
c(B) = \sum_{p=0}^{+\infty} c_p B^p \text{ is such that } \sum_{p=0}^{+\infty} p |c_p| < +\infty
\]
Decomposition into a persistent and a transitory component for processes integrated at seasonal frequencies

Under Condition $A$, there exists at frequency $\omega$ a Beveridge-Nelson type decomposition of the polynomial $c(B)$ as follows

$$c(B) = c(e^{i\omega}) + \tilde{c}_\omega(B) \delta_\omega(B)$$

where $\tilde{c}_\omega(B) = \sum_{p=0}^{+\infty} \tilde{c}_\omega,p B^p$ and $\tilde{c}_\omega,p = -\sum_{q=p+1}^{+\infty} c_k e^{i\omega(p-q)}$ such that $\{\tilde{c}_\omega(B) \varepsilon_t\}$ is a (complex) covariance stationary process.
Decomposition into a persistent and a transitory component for processes integrated at seasonal frequencies

\[ \delta_\omega S_\omega = I \] gives

\[ \delta_\omega (B) \left( x_t - c (e^{i\omega}) S_\omega \epsilon_t - \tilde{c}_\omega (B) \epsilon_t \right) = 0 \]

therefore

\[ x_t = \mu_t + c (e^{i\omega}) S_\omega \epsilon_t + \tilde{c}_\omega (B) \epsilon_t \]

where \( \mu_t = \mu e^{-i\omega t} \) is a deterministic function such that \( \delta_\omega (B) \mu_t = 0 \).

- An \( I_\omega (1) \) component: \( (\mu_t + c (e^{i\omega}) S_\omega \epsilon_t) \)
- A covariance stationary one \( (\tilde{c}_\omega (B) \epsilon_t) \)
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   - Integral operator algebra
   - Persistent components
3. Seasonal adjustment by subtraction of the seasonally persistent components
   - Introductory example: the Airline model
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   - Unit root testing
   - Demodulation and long term persistent component
5. Illustration
   - Monthly time series
   - Quarterly time series
Data generating process under study:

\[ y_t = d_t + x_t \]

where \( d_t \) is a deterministic component which is a linear combination of deterministic functions \( \{ e^{i\nu_j t} \}_{\nu_j \in \Omega_d} \) with \( \Omega_d = \{ \nu_0, \ldots, \nu_l \} \) and \( \{ x_t \} \) is an integrated of order one process at various frequencies in \( \Omega_x = \{ \omega_0, \ldots, \omega_k \} \).

\( \Omega_d \) and \( \Omega_x \) are subsets of the set \( \{ \tilde{\omega}_j = \frac{2j\pi}{s}, j \in \{0, 1, \ldots, s\} \} \) of seasonal frequencies associated to the seasonal length \( s \).
The airline model (Box and Jenkins (1976)) is given by

\[(1 - B)(1 - B^s) y_t = (1 - \theta B)(1 - \Theta B^s) \varepsilon_t\]

- \(s = 4\)

- By (1) and the commutativity property

\[(1 - B)(1 - B^4) S_0^2 S_\pi S_\frac{\pi}{2} S_{-\frac{\pi}{2}} = 1\]
\[ \forall t \in \mathbb{Z}, \ (1 - B) (1 - B^4) \left[ y_t - (1 - \theta B) (1 - \Theta B^4) S_0^2 S_{\pi} S_{\frac{\pi}{2}} S_{-\frac{\pi}{2}} \varepsilon_t \right] = 0 \]

There exists

\[ \mu_t = \mu_{0,2}^t + \mu_{0,1} + \mu_{\pi,1} (-1)^t + \mu_{\frac{\pi}{2},1} e^{i \frac{\pi}{2} t} + \mu_{\frac{\pi}{2},1} e^{-i \frac{\pi}{2} t} \]

such that

\[ y_t = \mu_t + (1 - \theta B) (1 - \Theta B^4) S_0^2 S_{\pi} S_{\frac{\pi}{2}} S_{-\frac{\pi}{2}} \varepsilon_t \]
By (3),

\[
S_0^2 S_\pi S_{\frac{\pi}{2}} S_{-\frac{\pi}{2}} = \frac{1}{4} \left( S_0^2 + \frac{3}{2} S_0 + \frac{1}{2} S_\pi + \frac{1}{1-i} S_{\frac{\pi}{2}} + \frac{1}{1+i} S_{-\frac{\pi}{2}} \right)
\]

\[
y_t = \mu_{0,2} t + \mu_{0,1} + \frac{1}{4} (1 - \theta B) (1 - \Theta B^4) \left( S_0^2 + \frac{3}{2} S_0 \right) \varepsilon_t
\]

\[
+ \mu_{\pi,1} (-1)^t + \frac{1}{8} (1 - \theta B) (1 - \Theta B^4) S_\pi \varepsilon_t
\]

\[
+ \mu_{\frac{\pi}{2},1} e^{i \frac{\pi}{2}} t + \frac{1}{4} (1 - \theta B) (1 - \Theta B^4) \frac{1}{1-i} S_{\frac{\pi}{2}} \varepsilon_t
\]

\[
+ \mu_{\frac{\pi}{2},1} e^{-i \frac{\pi}{2}} t + \frac{1}{4} (1 - \theta B) (1 - \Theta B^4) \frac{1}{1+i} S_{-\frac{\pi}{2}} \varepsilon_t
\]
Using the following expansions:

\[(1 - \theta B) (1 - \Theta B^4) = (1 + \theta) (1 - \Theta) + \theta \pi (B) (1 + B)\]

with

\[\theta \pi (B) = (-\theta + \Theta (\theta + 1) (1 - B + B^2 - B^3) + \Theta \theta B^4)\]

and

\[(1 - \theta B) (1 - \Theta B^4) = (1 + i\theta) (1 - \Theta) + \theta \pi (B) (1 + iB)\]

with

\[\theta \pi (B) = \Theta (1 - i\theta) (1 - iB - B^2 + iB^3) + i\theta (1 - \Theta B^4)\]
The Airline Model

we propose to limit the seasonal components to their pure non-stationary part

\[ y_t = y_s^t + y_t^* \]

with

\[ y_t^* = \mu_{0,2} t + \mu_{0,1} + \frac{1}{4} (1 - \theta B) (1 - \Theta B^4) \left( S_0^2 + \frac{3}{2} S_0 \right) \varepsilon_t + \theta (B) \varepsilon_t \]

where

\[ \theta (B) = \frac{1}{8} \left[ 3 ((1 + \theta) \Theta - \theta) + \Theta (1 - 3 \theta) B - (1 + \theta) \Theta B^2 + \Theta (\theta - 3) B^3 + 3 \Theta \theta B^4 \right] \]

and

\[ y_s^t = \mu_{\pi,1} (-1)^t + \frac{1}{8} (1 + \theta) (1 - \Theta) S_{\pi} \varepsilon_t \]

\[ + \mu_{\frac{\pi}{2},1} e^{i \frac{\pi}{2} t} + \mu_{\frac{\pi}{2},1} e^{-i \frac{\pi}{2} t} + \frac{1 - \Theta}{4} \left( \frac{(1 - i \theta)}{1 - i} S_{\pi} + \frac{(1 + i \theta)}{1 + i} S_{-\frac{\pi}{2}} \right) \varepsilon_t \]
The Airline Model

Airline model: $\theta=0.5, \Theta=0.1$
Seasonal random walks and demodulated Seasonal random walks
Persistence at seasonal frequency
Seasonal adjustment by subtraction of the seasonally persistent components
Practical implementation in a SARIMA framework
Illustration

The Airline Model

Airline model: $\theta = 0.5, \Theta = 0.1$
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Illustration

Introductory example: the Airline model
Univariate processes

DGP

\[
\begin{align*}
    y_t &= d_t + x_t \\
    d_t &= \sum_{j=0}^{l} \mu_j e^{iv_j t} \\
    \Delta_x(B)x_t &= c(B)\epsilon_t
\end{align*}
\]

where

\[
\Delta_x = \prod_{\omega \in \Omega_x} \delta_{\omega}
\]

and \(c(B)\) satisfies Condition A
From (1), we know that

$$\Delta_x (B) \prod_{j=0}^{k} S_{\omega_j} = 1$$

but

$$\Delta_x (B) \left( x_t - c (B) \left( \prod_{j=0}^{k} S_{\omega_j} \right) \varepsilon_t \right) = 0$$

so that

$$x_t = \mu_{x,t} + c (B) \left( \prod_{j=0}^{k} S_{\omega_j} \right) \varepsilon_t$$

where $$\mu_{x,t} = \sum_{j=0}^{k} \mu_{x,\omega_j} e^{-i\omega_j t}$$
Furthermore, we have

\[
\prod_{j=0}^{k} S_{\omega_j} = \sum_{j=0}^{k} \phi_j S_{\omega_j}
\]

with

\[
\phi_j = \frac{1}{\Delta x, -j (e^{i\omega_j})}
\]
For each frequency

\[ c (B) S_{\omega} \varepsilon_t = c (e^{i\omega}) S_{\omega} \varepsilon_t + \tilde{c}_{\omega} (B) \varepsilon_t \]

\[ y_t = y_t^* + y_t^s \]

with

\[ y_t^* = \mu_v_0 + \mu_x,\omega_0 + \phi_0 c (B) S_{0} \varepsilon_t + \sum_{j=1}^{k} \phi_j \tilde{c}_{\omega_j} (B) \varepsilon_t \]

and

\[ y_t^s = \sum_{j=1}^{l} \mu_v_j e^{i\omega_j t} + \sum_{j=1}^{k} \mu_x,\omega_j e^{-i\omega_j t} + \sum_{j=1}^{k} \phi_j c (e^{i\omega_j}) S_{\omega} \varepsilon_t \]
Implicit filter

\[ \Psi(B) = 1 - \sum_{j=1}^{k} \frac{\Delta_{x,-j}(B)}{\Delta_{x,-j}(e^{i\omega})} \frac{c(e^{i\omega_j})}{c(B)} \]

\[ \Psi(0) = 1 \text{ and close to 1 in its neighborhood} \]

Pseudo-spectrum of the “seasonally adjusted” process \( \{y_t^* - \mu_{t_0}\} \) is equal to

\[ f_{y^*}(\omega) = \left| \frac{1}{\Delta_{x,-0}(1)} \frac{c(e^{-i\omega})}{1 - e^{-i\omega}} + \sum_{j=1}^{k} \frac{\tilde{c}_{\omega_j}(e^{-i\omega})}{\Delta_{x,-j}(e^{i\omega})} \right|^2 \frac{\sigma_{\varepsilon}^2}{2\pi} \]
Spectral analysis

Phase shift and gain of the filter cannot be straightforwardly used to analyze the properties of the filter: applied on non-stationary processes

Example

DGP :\((1 - B^4)y_t = \epsilon_t\) then

\[ y_t = \frac{1}{4} \left( S_0 + S_\pi + S_{\frac{\pi}{2}} + S_{-\frac{\pi}{2}} \right) \epsilon_t \]

and

\[ y_t^* = \frac{1}{4} \left( 1 + B + B^2 + B^3 \right)y_t = \frac{1}{4} S_0 \epsilon_t \]

Grether and Nerlove (1970)'s comment: 'It has never been denied, and, indeed, is repeatedly emphasized by Rosenblatt in his own work that the effects noted in the frequency domain are significant only to the degree to which these same effects, translated into the behavior of the adjusted series over time, affect the interpretation of the movements of that series'.
Practical remarks

- annual sum discrepancy: for standard models, it is a annual white noise

\[
y_t^s = \sum_{j=1}^{l} \mu_v e^{iv_j t} + \sum_{j=1}^{k} \mu_{x,\omega_j} e^{-i\omega_j t} + \sum_{j=1}^{k} \phi_j c (e^{i\omega_j}) S_{\omega_j} \varepsilon_t
\]

\[(1 + B + \ldots + B^{s-1})y_t^s \text{ satisfies a MA}(q) \text{ with } q < s\]

- Computation of asymmetric filters for the beginning and the end of the sample: alternative to Musgrave method, Minimisation of the Mean Square Revision method, Best Linear Unbiased Estimates (BLUE) method, and Kenny and Durbin method.

- No over adjustment at the seasonal frequencies
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   - Quarterly time series
A first approach: testing simultaneously for the presence of all the unit roots that are associated to the seasonal difference operator equal to $1 - L^s$ (when the seasonal length is $s$). (Dickey, Hasza and Fuller (1984), Breitung and Franses (1998), Tanaka (1996)) → Taylor (2003) shows that this approach may lead to accept the seasonal unit root null hypothesis whereas the data generating process admits a unit root at the zero but not at seasonal frequencies.

A second approach: testing for the presence of each seasonal unit root (or couple of conjugate unit roots as we work with real time series). Asymptotic results for $\omega \in \{0, \pi\}$ and for $\omega \in ]0, \pi[$ differ (Hylleberg, Engle, Granger and Yoo (1991), Beaulieu and Miron (1993), Ghysels, Lee and Noh (1994), Rodrigues (2002), Gregoir (2006)
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Practical implementation in a SARIMA framework
Illustration

Unit root testing
Demodulation and long term persistent component

Computation of the long-run forecast of the demodulated process

Demodulation operation: \( \{ e^{i\omega t} x_t \} \) where \( \delta_{\omega} (B) x_t = c(B) \varepsilon_t \) and
\( x_t = \mu e^{-i\omega t} + c(e^{i\omega}) S_\omega \varepsilon_t + \tilde{c}_\omega (B) \varepsilon_t \)

\[
e^{-i\omega t} \lim_{h \to +\infty} E_t e^{i\omega(t+h)} x_{t+h} = \lim_{h \to +\infty} E_t e^{i\omega h} x_{t+h}
\]

\[
\lim_{h \to +\infty} E_t e^{i\omega h} x_{t+h} = \mu e^{-i\omega t} + c(e^{i\omega}) \sum_{\tau=1}^{t} e^{-i\omega(t-\tau)} \varepsilon_{\tau}
\]

\[
+ \lim_{h \to +\infty} E_t \left( c(e^{i\omega}) \sum_{\tau=t+1}^{t+h} \varepsilon_{\tau} e^{-i\omega(t-\tau)} + e^{i\omega h} \tilde{c}_\omega (B) \varepsilon_{t+h} \right)
\]

\[
= \mu e^{-i\omega t} + c(e^{i\omega}) S_\omega \varepsilon_t + \lim_{h \to +\infty} e^{i\omega h} E_t \tilde{c}_\omega (B) \varepsilon_{t+h}
\]

\( l_\omega(1) \) component goes to 0

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An alternative framework for univariate and multivariate seasonal adjustment
Computation of the long-run forecast of the demodulated process

For $t \geq 0$:

$$
\lim_{h \to +\infty} E_t e^{i\omega(t+h)} x_{t+h} = \mu \omega + c \left( e^{i\omega} \right) \sum_{\tau=1}^{t} e^{i\omega \tau} \varepsilon_{\tau}
$$

but

$$
e^{i\omega(t+h)} x_{t+h} = e^{i\omega t} x_t + \sum_{\tau=1}^{h} \left( e^{i\omega(t+\tau)} x_{t+\tau} - e^{i\omega(t+\tau-1)} x_{t+\tau-1} \right)
$$

$$
= e^{i\omega t} x_t + \sum_{\tau=1}^{h} e^{i\omega(t+\tau)} \delta_{\omega} (B) x_{t+\tau}
$$
Computation of the long-run forecast of the demodulated process

\[
\mu_\omega + c \left( e^{i\omega} \right) \sum_{\tau=1}^{t} e^{i\omega \tau} \varepsilon_\tau = e^{i\omega t} x_t + \sum_{\tau=1}^{+\infty} e^{i\omega (t+\tau)} E_t \delta_\omega (B) x_{t+\tau}
\]

by demodulation

\[
\mu_\omega e^{-i\omega t} + c \left( e^{i\omega} \right) S_\omega \varepsilon_t = x_t + \sum_{\tau=1}^{+\infty} e^{i\omega \tau} E_t \delta_\omega (B) x_{t+\tau}
\]
Univariate SARIMA process

DGP

\[ \Phi(B) \Phi_s(B^s) \Delta_x(B) x_t = \Theta(B) \Theta_s(B^s) \varepsilon_t \]

where \( d^0\Phi = P \), \( d^0\Phi_s = P_s \), \( d^0\Theta = Q \), \( d^0\Theta_s = Q_s \) and all the roots of these polynomials have a modulus strictly larger than 1. We consider the \((P + sP_s + Q + sQ_s) \times 1\) process

\[
\begin{align*}
Z_t &= \begin{pmatrix}
\Delta_x(B)x_t & \ldots & \Delta_x(B)x_t-(P+sP_s) & \varepsilon_t & \ldots & \varepsilon_t-(Q+sQ_s)
\end{pmatrix}' \\
\zeta_t &= \begin{pmatrix}
\varepsilon_t & 0 & \ldots & 0 & \varepsilon_t & 0 & \ldots & 0
\end{pmatrix}'
\end{align*}
\]

and the usual \((P + sP_s + Q + sQ_s) \times (P + sP_s + Q + sQ_s)\) companion matrix \( C \) associated to this process such that

\[ Z_t = C Z_{t-1} + \zeta_t \]
Univariate SARIMA process

\[ E_t z_{t+\tau} = C^\tau z_t \]

whence if \( K = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix} \) then

\[ K' \sum_{\tau=1}^{+\infty} e^{i\omega\tau} E_t \Delta x(B) x_{t+\tau} = K' \left( \sum_{\tau=1}^{+\infty} e^{i\omega\tau} C^\tau \right) z_t \]

\[ = K' \left( I_{P+sP_s+Q+sQ_s} - e^{i\omega C} \right)^{-1} e^{i\omega C} z_t \]

\[ x_t^s = \sum_{j=1}^{k} \frac{\Delta x_{-j} x_t + K' \left( I_{P+sP_s+Q+sQ_s} - e^{i\omega j C} \right)^{-1} e^{i\omega j C} z_t}{\Delta x_{-j} \left( e^{i\omega j} \right)} \]
Outline

1. Seasonal random walks and demodulated Seasonal random walks
2. Persistence at seasonal frequency
   - Integral operator algebra
   - Persistent components
3. Seasonal adjustment by subtraction of the seasonally persistent components
   - Introductory example: the Airline model
   - Univariate processes
4. Practical implementation in a SARIMA framework
   - Unit root testing
   - Demodulation and long term persistent component
5. Illustration
   - Monthly time series
   - Quarterly time series
### Table: Test for seasonal unit root

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Test statistic</th>
<th>5%–level</th>
<th>Decision</th>
</tr>
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<tbody>
<tr>
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<td>$\pi$</td>
<td>1.29</td>
<td>3.11</td>
<td>Rejection of $H_0$</td>
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French Household consumption in manufactured goods

Household consumption in manufactured goods
1980-1997

- n.s.a time series
- new method s.a. time series
Seasonal random walks and demodulated seasonal random walks
Persistence at seasonal frequency
Seasonal adjustment by subtraction of the seasonally persistent components
Practical implementation in a SARIMA framework
Illustration

Spectral analysis

spectral coherency n.s.a time series - s.a. INSEE

spectral coherency n.s.a time series - new method s.a.

phase shift n.s.a. time series - s.a. INSEE

phase shift n.s.a. time series - new method s.a.
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Illustration

Turning points

Household consumption in manufactured goods
1980-1997

f irst turning point

second turning point

third turning point

fourth turning point

INSEE s.a.
s.a. new method

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An alternative framework for univariate and multivariate seasonal...
Seasonal random walks and demodulated Seasonal random walks
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Growth rate

Household consumption in manufactured goods
growth rate 1980-1997

INSEE s.a.
s.a. new method

Stéphane Gregoir
An alternative framework for univariate and multivariate seasonal
Japanese Private Consumption

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<table>
<thead>
<tr>
<th>Frequency</th>
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<th>decision</th>
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Monthly time series
Quarterly time series

Japanes private consumption
1980-2003 (Q3)

n.s.a.
Japanese Institute s.a.
new s.a.

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Japanese private consumption growth rate 1980-2003 (Q3)

Japanese Institute s.a.
new s.a.

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spectral coherency n.s.a. time series - s.a. NIS
spectral coherency n.s.a time series - new s.a.

phase shift n.s.a time series - s.a. NSI
phase shift n.s.a. time series - new s.a.

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An alternative framework for univariate and multivariate seasonal
Conclusion

- Simple linear method
- Estimation of the weak linear representation can be done with robust methods (less sensitive to outliers)
- Gives a rationale for the computation of asymmetric filters at the end of the sample (DGP dependent)
- Evolving calendar effect (working days, lunar holidays) can be treated in the same framework
- Extension to locally-stationary processes