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Tommaso Proietti

Dip. Di Scienze Statistiche, University of Udine and
Department of Economics, European University Institute

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Tommaso Proietti*
Dip. di Scienze Statistiche, University of Udine and
Department of Economics, European University Institute

Preliminary

Abstract

This paper discusses several issues related to trend-cycle decompositions with correlated components of macroeconomic time series, and illustrates them with reference to the Euro area and the Italian gross domestic product. In particular, we address the small sample properties of the estimated correlation of the trend and cycle disturbances, and review the interpretative issues raised by these decomposition.

The nature of inferences about trends and cycles, with reference to the real time and final estimates, and the related topic of revision, is considered, along with the relationship with other popular results, such as the Beveridge and Nelson decomposition, the Single Source of Error and the Innovation models.

We also look at the consequences of seasonal adjustment and temporal aggregation on the empirical evidence for a negative correlation between the disturbances. Finally, we illustrate that multivariate analysis can provide additional insight on this topic.

Keywords: Deviation and Growth Rate cycles; Hysteresis; Kalman Filter and Smoother; Innovation Models; Seasonal Adjustment; Temporal Aggregation.

*Address for correspondence: Dept. of Economics, European University Institute - Badia Fiesolana, Via dei Roccettini 9, San Domenico di Fiesole (FI) I-50016 Tel. +39 055 4685 759. E-mail: tproiett@iue.it

1 Introduction

This paper is concerned with unobserved components (UC) models for the decomposition of a macroeconomic aggregate into a trend component and a *deviation* cycle. Unobserved components (UC) models assume that the components are driven by orthogonal disturbances (see, for instance, Clark, 1987, Harvey and Jäger, 1994) or perfectly correlated ones; in the latter case there is a single source of disturbances and the sign of the correlation is implied by the remaining parameter estimates. These restrictions are often enforced to produce just-identified decompositions, but in some cases they are over-identifying.

When there are no degrees of freedom available for estimating the correlation between the disturbances, and economic theory is uninformative about this parameter, we cannot usually discriminate between different assumptions by the usual likelihood inferences, although departures from the maintained model will show up in the several diagnostic tools in the econometrician kit. The reduced form, i.e. the corresponding model in the ARIMA class, will be of the same order, but models with orthogonal disturbances impose severe restrictions on its parameter space.

To overcome the latter, models with correlated components have been considered by Godolphin (1976) and Godolphin and Stone (1980), with the explicit intent of extending the parameter range yielding decomposable models. Snyder (1985), Ord, Koehler and Snyder (1997), and Hyndman *et al.* (2002) advocate state space models with only one source of random disturbances, with the same intent, also arguing that inferences are simplified. Another very popular result, the Beveridge and Nelson (BN, 1981) decomposition, is formulated in terms of perfectly correlated disturbances, and is commonly viewed as providing a structural interpretation to *any* ARIMA model. In the study of macroeconomic time series (e.g. GDP at constant prices) the common disturbance has often been associated with productivity, or real, shocks. The BN decomposition is actually a particular case of what is known as a *formal* decomposition of an ARIMA model; see Brewer *et al.* (1975), Brewer (1979) and Piccolo (1982). Casals, Jerez and Sotoca (2002) have recently advocated the use of the innovation form of a structural model for inference about unobserved components. Their argument is that this representation yields an "exact" decomposition, such that the model for the estimated component is congruent with the theoretical one and the components are estimated in real time, i.e. using only current and past information.

Models with perfectly correlated disturbances pay a price for their wider applicability: at the outset there is no guarantee that the components will be sensible. For instance, we must be willing to accept that trend growth has higher unconditional variance than output growth, which may be regarded as quite implausible, under a weak smoothness prior. Moreover, if main motivation for entertaining them was to enlarge the reduced form decomposable parameter range, but they happened to select a point in that space for which an orthogonal decomposition is admissible, the estimation of the components could have been improved by using future observations; smoothing is however prevented by the model specification itself.

In other words, the restrictions imposed by UC with uncorrelated components are often reasonable, providing plausible ways of weighting the data and of avoiding, for instance, that the trend fluctuates more wildly than the observations. Harvey and Koopman (2000), looking at the implications on the weighting patterns for signal extraction, cast some doubts on the plausibility of models with correlated components.

These issues have been reflected in the forecasting literature. The "structural" interpretation of the forecast function of any ARIMA model has been provided by Box, Pierce and Newbold (1987), using a partial fraction expansion of the autoregressive polynomial; by this approach, which is essentially the same as that at the foundation of the BN decomposition, it is possible to derive updating equations for the components of the forecast function that depend on the innovations. The fundamental question is whether the components can be genuinely interpreted as trends, cycles, etc., especially when the innovations are not discounted; see Proietti (2002a, sec. 5.10) for an illustration. On the other hand, UC models with orthogonal disturbances impose some kind of discounting on the innovations that is coincident with or in the same spirit of that arising in exponential smoothing techniques. With reference to the latter, the related problem as to whether smoothing constant greater than 1 are admissible has received some attention, and is reviewed in Gardner (1985).

A different situation arises when the restrictions on the correlation are over-identifying. This occurs if the representation chosen for the components is not "saturated": usually, UC models are a linear combinations of individual components, such that the i-th component has a (possibly nonstationary) ARMA$(p_i, p_i - 1)$ representation. For instance, a typical trend-cycle decomposition features a random walk (RW) trend ($p_1 = 1$) plus a stationary ARMA(2,1) cycle ($p_2 = 2$). If the parameters are unconstrained, the model is "saturated" and we have to assume a particular value for the correlation of the disturbances driving the components so as to achieve exact identification. If the cycle is specified as a pure AR(2), instead, the reduced form has one more parameter than the UC model and this extra degree of freedom can be used to estimate the correlation between the trend and cycle disturbances.

Morley, Nelson and Zivot (2002, MNZ henceforth) have recently contributed to this issue: they consider a class of UC decompositions of U.S. real gross domestic product (GDP) into a random walk trend and a purely AR(2) cycle, that depends on the identifiable correlation between the trend and cycle disturbances and that produces an ARIMA(2,1,2) reduced form. Within this class, MNZ compare the fit and the components arising from the UC model assuming orthogonal disturbances and the BN decomposition of the unrestricted ARIMA model, which features perfectly and negatively correlated disturbances. The resulting decompositions produce different stylised facts, and in particular the BN cycle is characterised by a much smaller amplitude and a shorter periodicity.

Since a degree of freedom is allowed from the fact that the UC model has one parameter less than the ARIMA reduced form, they estimate the correlation between the trend and cycle disturbances and find out that the estimated value is negative, about -0.92, and significantly different from zero. The resulting *real time*, or concurrent, estimates of the trend and cycle in U.S. GDP closely resemble the BN components, which allows us to reconcile the UC with the unrestricted reduced form.

They interpret this empirical evidence as an expression of the dominant role of real shocks, which shift the long run path of output, whereas short term fluctuations reflect only the adjustment to the new path.

This paper will be concerned with the estimation and the interpretation of decomposition with correlated trend and cycle disturbances. We set up with the specification of the benchmark UC model in section 2, which nests two leading cases of interest: the UC model with correlated disturbances and the orthogonal decomposition with ARMA(2,1) cyclical component. Section 3 reviews the BN decomposition of the ARIMA(2,1,2) reduced form and its main properties, while section 4 presents the autocovariance and spectral generating functions of the various models.

In section 5 we illustrate and compare the fit of orthogonal and correlated UC models to the Euro area and the Italian quarterly real gross domestic product. The evidence for the Italian series mirrors quite closely the findings by MNZ, i.e. in favour of a strong and negative correlation between the trend and cycle disturbances, although we argue that the small sample distribution of the correlation coefficient raises some concern. For the Euro area the findings are not conclusive.

Models with correlated components pose several interpretative issues since, under certain conditions, they are observationally equivalent to models that provide different explanations of the nature of macroeconomic fluctuations (section 6). Strongly and negatively correlated disturbances imply that the spectral density of the first differences of the series is not a global minimum at the long run frequency. This feature is accommodated also by the cyclical growth model, that can also be parameterised as a model featuring hysteresis effects. The Italian case illustrates that the cyclical growth model and the hysteresis model possess exactly the same explanatory power, yielding the same likelihood.

Inference about unobserved components in models with correlated components is dealt with section 7, where we consider the state space representation, the treatment of initial conditions, estimation of the components in real time and using the full sample, and some of the ambiguities that arise for single source of error and innovation form representation when they are considered as "models". We also illustrate that a very peculiar trait of models with highly and negatively correlated trend and cycle disturbances is that the future is more informative than the past for signal extraction. As a result the cycle estimates will be subject to large revisions and the final estimates will display greater amplitude than the real time ones.

Sections 9 and 10 investigate respectively whether seasonal adjustment and temporal aggregation can affect the empirical evidence about the sign and the magnitude of the correlation coefficient. Finally, we address the issue as to whether multivariate UC models can cast some light on correlated disturbances (section 11). Section 12 draws the main conclusions.

2 Trend-Cycle decomposition with Correlated Components

The basic univariate representation for an output series, y_t, deals with the decomposition into a random walk trend component, denoted μ_t, and a stationary ARMA(2,1) stochastic cycle, denoted ψ_t:

$$\begin{aligned}
y_t &= \mu_t + \psi_t \qquad\qquad\qquad\qquad t = 1, 2, \ldots, T, \\
\mu_t &= \mu_{t-1} + \beta + \eta_t, \\
\psi_t &= \phi_1 \psi_{t-1} + \phi_2 \psi_{t-2} + \kappa_t + \theta \kappa_{t-1}, \qquad (1) \\
\begin{pmatrix} \eta_t \\ \kappa_t \end{pmatrix} &\sim \text{NID}\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_\eta^2 & \sigma_{\eta\kappa} \\ \sigma_{\eta\kappa} & \sigma_\kappa^2 \end{pmatrix}\right], \qquad \sigma_{\eta\kappa} = r\sigma_\eta\sigma_\kappa.
\end{aligned}$$

The trend and cycle disturbances are allowed to be contemporaneously correlated, with r being the correlation coefficient; NID denotes normally and independently distributed random variables. Complex stationary autoregressive roots can be imposed expressing $\phi_1 = 2\rho\cos\lambda_c$ and $\phi_2 = -\rho^2$, where ρ and λ_c (representing the modulus and the phase of the roots of the AR characteristic equation), lie respectively in $[0, 1)$ and $[0, \pi]$.

Model (1) will be labelled UC(r, θ) to stress the dependence on the two "conflicting" parameters. Its reduced form is the ARIMA(2,1,2) process:

$$\Delta y_t = \beta + \frac{\theta(L)}{\phi(L)}\xi_t, \quad \xi_t \sim \text{WN}(0, \sigma^2), \qquad t = 2, ..., T, \qquad (2)$$

where $\theta(L) = 1 + \theta_1 L + \theta_2 L^2$ and $\phi(L) = 1 - \phi_1 L - \phi_2 L^2$ are respectively the MA and AR polynomials in the lag operator, L, and $\Delta = 1 - L$.

The reduced form has six parameters, whereas UC(r, θ) has seven. Hence, the latter is not identified and one has to restrict either r or θ. The orthogonal trend cycle decomposition considered by Clark (1987) imposes $r = \theta = 0$, and thus will be denoted UC(0,0). MNZ entertain UC$(r, 0)$ and compare it with UC$(0, 0)$. Harvey and Jäger (1994), although they entertain I(2) - local linear - trends, consider UC$(0, \theta)$, with a restricted θ, which functionally depends on ρ and λ_c.

It should be noticed that UC$(r, 0)$ is not identifiable if $\phi_2 = 0$; the parameterisation in terms of the modulus and phase of the AR process avoids this lack of identification if $0 < \rho < 1$.

3 The Beveridge-Nelson Decomposition

The Beveridge and Nelson (1981) decomposition hinges upon the definition of the trend in terms of prediction as the value at time t of the eventual forecast function. When y_t is difference stationary, we can uniquely decompose the series process into a random walk trend and a stationary transitory component. The decomposition has been deemed to provide a structural interpretation to any ARIMA$(p, 1, q)$ reduced form model fitted according to the traditional Box-Jenkins methodology, with the characterising property that the components are driven by perfectly correlated disturbances, that are linear in the innovations, ξ_t.

With respect to the ARIMA(2,1,2) model (2), the BN decomposition specialises as follows:

$$y_t = m_t + c_t, \qquad t = 1, ..., T. \qquad (3)$$

where the trend, m_t, has the random walk representation:

$$m_t = m_{t-1} + \beta + \frac{\theta(1)}{\phi(1)}\xi_t. \tag{4}$$

and the cycle, c_t, has the ARMA(2,1) representation:

$$\phi(L)c_t = (1 + \vartheta^* L)\left[1 - \frac{\theta(1)}{\phi(1)}\right]\xi_t, \qquad \vartheta^* = -\frac{\phi_2\theta(1) + \theta_2\phi(1)}{\phi(1) - \theta(1)}. \tag{5}$$

These results follow straightforwardly from Proietti (1995) and Proietti and Harvey (2000).

It is apparent from (4) and (5) that the two components are driven by the innovations, ξ_t; the fraction $\theta(1)/\phi(1)$, known as *persistence*, is integrated in the trend, and its complement to 1 drives the cycle. The sign of the correlation between the trend and the cycle disturbances is provided by the sign of $\phi(1) - \theta(1)$; when persistence is less (greater) than one then trend and cycle disturbances are positively (negatively) and perfectly correlated.

The BN cycle has always an MA feature, unless $\phi_2\theta(1) + \theta_2\phi(1) = 0$. The MA polynomial can be non invertible, i.e. $|\vartheta_1^*|$ can be greater than 1; this will be the case for the ARIMA(2,1,2) models estimated in section 5 for the Euro Area and the Italian GDP.

As shown by Watson (1989) the BN components, defined on the reduced form of UC models, are always coincident with the filtered, or real time time, estimates arising from the UC(r, θ) model, whatever restriction we impose to make it identifiable. The filtered components of *identified* UC(r, θ) models are however estimated with non zero mean square error even in the case $r = -1, \theta = 0$. Hence, it would not be correct to regard the BN trend and cycle as the estimates of the components arising from UC(-1,0), as future observations reduce the estimation error. This point has often been overlooked in the literature and we return to it in section 8, where we show that the only case in which ψ_t is actually an observed component in real time arises for $r = 1$.

When the BN decomposition is interpreted *as a model*, the components are estimated in real time with zero mean square error, after processing a suitable number of observations so that the effect of initial conditions is marginalised, this being the only source of uncertainty (assuming known parameters). We discuss this further in section 7.

4 Autocovariance Generating Functions

The properties of any linear time series model of economic fluctuations are uniquely characterised by its autocovariance generating function (ACGF). The ACGF also provides a valuable tool to address the equivalence issues that arise in the interpretation of models with correlated disturbances (see section 6). Moreover, its frequency domain counterpart, the spectral generating function (SGF), will be used for estimating the parameters of the model by maximum likelihood.

The ACGF of the reduced form model for Δy_t, denoted $g(L)$, is $g(L) = \sigma^2|\theta(L)|^2/|\phi(L)|^2$, where $|\theta(L)|^2 = \theta(L)\theta(L^{-1})$ and $|\phi(L)|^2 = \phi(L)\phi(L^{-1})$. The various UC model that result by constraining (1) are restricted versions of $g(L)$. For the UC(r,0) model considered by MNZ the ACGF, denoted $g_r(L)$, can be written as follows (Proietti, 2002):

$$|\phi(L)|^2 g_r(L) = |\phi(L)|^2\sigma_\eta^2 + |1 - L|^2[\sigma_\kappa^2 + r\sigma_\eta\sigma_\kappa(1 + \phi_1 + \phi_2 + \phi_2(L + L^{-1}))]. \tag{6}$$

Equating $g(L)$ to $g_r(L)$ provides the way of deriving the reduced form parameters $(\theta_1, \theta_2, \sigma^2)$ from $(\sigma_\eta, \sigma_\kappa, r)$ and of assessing the restrictions imposed by the UC model on the reduced form. For instance, $g(1) = g_r(1)$ implies $\sigma^2_\eta = \sigma^2[\theta(1)/\phi(1)]^2$.

For the UC$(0, \theta)$ model we have

$$|\phi(L)|^2 g_\theta(L) = |\phi(L)|^2\sigma^2_{\eta^*} + |1 - L|^2|1 + \theta L|^2\sigma^2_{\kappa^*} \tag{7}$$

where, with a change of notation that will be useful in the sequel, $\sigma^2_{\eta^*}$ and $\sigma^2_{\kappa^*}$ denote the variance of the trend and cycle disturbances when we assume in (1) that they are mutually uncorrelated at all leads and lags.

The ACGF of the Clark model, UC(0,0), is obtained by setting $r = 0$ in (6) or $\theta = 0$ in (7). Replacing L with the complex exponential $e^{-\imath\lambda} = \cos\lambda - \imath\sin\lambda$, where $\imath$ is the imaginary unit, gives the spectral generating function, that provides a decomposition of the variance of Δy_t into the contribution of changes in the trend, in the cycle and, in the case of UC(r,0), the covariation (cross spectral density) between the two.

5 Two Illustrative Examples

This section illustrates the fit of the unrestricted ARIMA(2,1,2) and three different trend-cycle decompositions that result from (1), UC(0,0), UC$(r, 0)$ and UC$(0, \theta)$, with respect to Euro Area (EA) and the Italian Gross Domestic Product (GDP) at constant prices. Both series are quarterly and are available for the sample period 1970:1-2002.2. The EA series is an update of the one constructed for the Area Wide Model by Fagan, Henry and Mestre (2001), and the Italian series is made available electronically at www.istat.it.

Model estimation has been carried out in the frequency domain. The likelihood is defined in terms of the stationary representation of the various models, that is in terms of $\Delta y_t, t = 1, \dots, T^* = (T - 1)$; see Nerlove, Grether and Carvalho (1995) and Harvey (1989, sec. 4.3). While the time domain likelihood of UC models is based on a recursive orthogonalisation, known as the prediction error decomposition, performed by the Kalman filter (section 7), the frequency domain one is based on an alternative orthogonalisation, achieved through a Fourier transform. Denoting the Fourier frequencies by $\lambda_j = \frac{2\pi j}{T^*}$, $j = 0, 1, \dots, (T^* - 1)$, the likelihood function is defined as follows:

$$\text{loglik} = -\frac{1}{2}\left\{T^*\log 2\pi + \sum_{j=0}^{T^*-1}\left[\log g_m(\lambda_j) + 2\pi\frac{I(\lambda_j)}{g_m(\lambda_j)}\right]\right\}$$

where $g_m(\lambda_j) = g_m(e^{-\imath\lambda_j})$ denote the spectral generating function of the m-th model evaluated at frequency λ_j, and $I(\lambda_j)$ is the periodogram:

$$I(\lambda_j) = \frac{1}{2\pi}\left[c_0 + 2\sum_{\tau=1}^{T^*-1} c_\tau\cos(\lambda_j\tau)\right]$$

where c_τ denotes the sample autocovariance at lag τ,

$$c_\tau = \frac{1}{T^*}\sum_{t=1}^{T-\tau}(\Delta y_t - \bar{\Delta y})(\Delta y_{t-\tau} - \bar{\Delta y}), \quad \bar{\Delta y} = \frac{1}{T^*}\sum_{t=1}^{T^*}\Delta y_t.$$

The index m refers alternatively to the ARIMA model, UC(0,0), UC(r,0), and UC($0,\theta$). The corresponding spectral generating functions are straightforwardly derived from the ACGFs presented in section 4[1].

Table 1 presents the main estimation results along with some diagnostics: $Q(12)$ denotes the Ljung-Box portmanteau test statistic for residual autocorrelation based on the first 12 autocorrelations, and we also present the Doornik and Hansen (1994) test of normality. Both are computed on the standardised Kalman filter innovations (see appendix 7).

Table 1: Parameter estimates and diagnostics for models of quarterly Euro Area and Italian GDP, 1970.1-2002.2; (r) denotes a restricted parameter.

	Euro Area				*Italy*			
	ARIMA	UC(0,0)	UC(r,0)	UC($0,\theta$)	ARIMA	UC(0,0)	UC(r,0)	UC($0,\theta$)
ϕ_1	1.40	1.65	1.40	1.65	1.47	1.54	1.47	1.56
ϕ_2	-0.69	-0.68	-0.69	-0.68	-0.77	-0.59	-0.77	-0.84
θ_1	-1.17				-1.14			
θ_2	0.57				0.48			
σ^2	0.3443				0.5225			
r		0(r)	-0.95	0(r)		0(r)	-0.82	0(r)
σ^2_η ($\sigma^2_{\eta^*}$)		0.2090	0.6473	0.2435		0.1475	0.6672	0.3957
σ^2_κ ($\sigma^2_{\kappa^*}$)		0.0975	0.2226	0.0350		0.3560	0.2539	0.0216
θ		0(r)	0(r)	0.64		0(r)	0(r)	1.00
loglik	-114.28	-114.52	-114.28	-114.59	-141.17	-144.04	-141.17	-143.03
$Q(12)$	6.83	7.70	6.83	7.60	6.29	13.53	6.29	14.07
Normality	11.80	11.63	11.80	11.64	2.08	1.53	2.08	2.46

We observe that for both series the ARIMA model and UC($r,0$) provide exactly the same likelihood inferences; hence the reduced form of the latter coincides with the unrestricted ARIMA(2,1,2) model fitted to the series. The persistence parameter is respectively 1.38 (EA) and 1.13 (Italy). The estimated correlation parameter is high and negative (-0.95 for EA and -0.82 for Italy), and the ratio $\sigma_\eta/\sigma_\kappa$ is always greater that 1. The likelihood ratio (LR) test of the restriction $r = 0$ is not significant for EA but it is highly so for Italy, with a p-value 0.02, whereas the LR test of $\theta = 0$ is never significant. It is noticeable that for Italy the estimated cycle MA parameter lies on the boundary of the parameter space; in general models with orthogonal disturbances yield worse Ljung-Box statistics.

The fit provided by the models to the periodogram (raw sample spectrum) emerges from figure 1, which presents $I(\lambda_j)$ along with the estimated spectral density functions $g_r(\lambda_j)/(2\pi)$ and $g_\theta(\lambda_j)/(2\pi)$. Obviously, $g_r(\lambda_j) = g(\lambda_j)$, that is the ARIMA spectrum is identical to that implied by UC(r,0).

For EA the spectral density fitted by UC(0,θ) is characterised by a spectral peak taking place at a lower frequency, and therefore the resulting cycle estimates are characterised by a larger period (the estimated AR polynomial can actually be written as $\phi(L) = (1-0.82L)^2$ and thus features a stationary root with multiplicity 2 at the zero frequency - this is a

[1] All the computations were performed in Ox 3.2 (Doornik, 2001). Signal extraction was performed by the Kalman filter and smoother using the library of state space function SsfPack 3.0 (beta) by Koopman et al. (1999), linked to Ox 3.2.

special case of a second order cycle, see Harvey and Trimbur, 2002); as expected, UC(r, 0), yields a much higher estimate at the zero frequency and it is not a minimum at that frequency. We also notice a periodogram ordinate close to the Nyquist frequency that is not fitted by any of the models: this corresponds to the effect of the calendar component in GDP (number of working/trading days in the quarter).

For Italy, the cycle periods are not different (about 3 years), although $g_r(\lambda_j)$ and $g_\theta(\lambda_j)$ differ around the zero frequency and the spectral peak. For the latter we observe that $\theta = 1$ implies $g_\theta(0) = g_\theta(\pi)$ as the cycle is strictly non invertible at the π frequency. The richer residual autocorrelation pattern characterising UC(0,θ) are likely to be a consequence of underestimation of the zero frequency variance component.

For estimation purposes, we adopted the transformation $r = \bar{r}/\sqrt{1+\bar{r}^2}$, where $\bar{r}$ is estimated unrestrictedly and the transformation ensures that the correlation parameter is constrained in the admissible range [-1,1]. The asymptotic standard error of r, estimated by the Delta method, are 0.17 and 0.22, respectively for the EA and Italy. These, however, provide only a very bad guidance over the sampling distribution of r. To illustrate this point we generated 1000 bootstrap estimates of r; the distribution is plotted in figure 2. For the implementation of the bootstrap we followed Stoffer and Wall (1991), generating 1000 series with the same sample size of the original ones by resampling without replacement the standardised innovations arising from the fitted UC(r,0) model. The sampling distribution is highly nonstandard as it suffers from a "pile-up" phenomenon at the extremes of the sample range; if we consider that in the Italian case 27% and 6% are equal respectively to -1 and +1, a boostrap confidence interval covers all the parameter range. The same considerations apply to EA.

In conclusion, the results presented in this section confirm the MNZ findings, pointing out that among the unobserved components models considered, the UC($r, 0$) model is the only one that can be reconciled with the unrestricted ARIMA(2,1,2) model of GDP. However, only for Italy the correlation between trend and cycle disturbances resulted significant using standard asymptotic inferences. The bootstrap characterisation of the sampling distribution of the correlation parameter suggests that those inferences need to be handled with great care.

6 Interpretative Issues

The evidence emerging from our empirical illustrations, would, with the caveats made above, point out in the direction of selecting the UC(r, 0) specification, with a high and negative correlation between the trend and cycle disturbances. This, along with the signal ratio $\sigma^2_\eta/\sigma^2_\kappa$ being relatively high, has been taken to support the notion of the prominence of real shocks, η_t, as opposed to nominal ones, κ_t.

In this section we review some alternative ways of interpreting the correlation between the disturbances, by establishing the conditions under which UC(r, 0) can be viewed as a reparameterisation of an alternative decomposition with a very different meaning.

6.1 The Equivalence of UC(r,0) and UC($0, \theta$)

The UC(r,0) model can be rewritten as an UC(0,θ) model if the quadratic equation,

$$\frac{\theta}{1+\theta^2} = \phi_2 r \frac{\sigma_\eta}{\sigma_\kappa} \left[1 + r\frac{\sigma_\eta}{\sigma_\kappa}(1+\phi_1+\phi_2)\right]^{-1}, \tag{8}$$

admits a real and invertible solution (Proietti, 2002). The remaining parameters are then obtained as follows: $\sigma^2_{\eta^*} = \sigma^2_\eta$ and $\sigma^2_{\kappa^*} = \phi_2 r \sigma_\eta \sigma_\kappa / \theta$. These results are derived form the ACGF identity, $g_\theta(L) = g_r(L)$, which amounts to equating the right hand sides in (6) and (7).

The admissibility conditions can be shown to be exactly the same under which $g_r(e^{-\imath\lambda})$ is a global minimum at the zero frequency. This reflects the fundamental fact that the orthogonal decomposition UC($0, \theta$) imposes that the spectral density of Δy_t is a minimum at zero, a result already established in Lippi and Reichlin (1992).

The equivalence is always feasible if r is positive, but, we can allow for negative correlation provided that the ratio $\sigma_\eta/\sigma_\kappa$ is small, i.e. the trend disturbance is a minor source of variation. In conclusion, when the spectral density of Δy_t is a minimum at zero, the cross spectrum between the components absorbs part of the cyclical variability; this can be reallocated to the cyclical component, which is underestimated by the UC(r,0) model, by allowing it to display a moving average feature.

6.2 Cyclical Growth and Hysteresis

Consider now the following UC model that postulates that Δy_t can be additively decomposed into a cyclical component and orthogonal noise:

$$\begin{array}{rcll} \Delta y_t & = & \beta + \psi_t + \eta^*_t, & \eta^*_t \sim \mathrm{WN}(0, \sigma^2_{\eta^*}), \\ \psi_t & = & \phi_1\psi_{t-1} + \phi_2\psi_{t-2} + \kappa^*_t + \theta\kappa^*_{t-1}, & \kappa^*_t \sim \mathrm{WN}(0, \sigma^2_{\kappa^*}), \\ \mathrm{E}(\eta^*_t \kappa^*_t) & = & 0. & \end{array} \tag{9}$$

The idea is that of representing underlying growth as a smooth cyclical process.

Model (9) has again an ARIMA(2,1,2) reduced form, and six parameters, but different implications. In its original specification, it simply produces estimates of underlying growth that are smoother than the original observations; it can also be interpreted as a *cyclical trend model*, as in Harvey (1989, p. 46), such that the trend is coincident with the observations, i.e. $y_t = \mu_t$ and $\mu_t = \mu_{t-1} + \beta + \psi_t + \eta^*_t$.

It is also observationally equivalent to the Jäger and Parkinson (1994) *hysteresis* model, which is such that a deviation cycle can still be defined, but the cycle modifies also permanently the trend. The hysteresis model is specified as follows:

$$\begin{array}{rcll} y_t & = & \mu_t + \psi_t, & t = 1, 2, \ldots, T, \\ \mu_t & = & \mu_{t-1} + (1+\theta)\psi^*_{t-1} + \eta^*_t, & \eta^*_t \sim \mathrm{WN}(0, \sigma^2_{\eta^*}), \\ \psi^*_t & = & \phi_1\psi^*_{t-1} + \phi_2\psi^*_{t-2} + \kappa^*_t, & \kappa^*_t \sim \mathrm{WN}(0, \sigma^2_{\kappa^*}) \end{array} \tag{10}$$

and $\mathrm{E}(\eta^*_t \kappa^*_t) = 0$. Notice that the cycle, ψ^*_t, is redefined as a pure second order AR process; $(1+\theta)$ represents the hysteresis parameter, i.e. the fraction of the cycle that is integrated in the trend. Obviously, $\theta = -1$ yields again the additive decomposition into orthogonal trend and cycle that corresponds to the Clark model UC$(0, 0)$.

Using the same expedient of equating the ACGFs, we establish a set of conditions under which (9) can provide a trend - cycle decomposition with correlated disturbances, i.e. can be written as an UC(r, 0) process. These are met if we can uniquely determine the cycle MA parameter θ in (9) for given values of the correlation parameter r and the ratio $\sigma_\eta/\sigma_\kappa$ in UC$(r, 0)$, as the admissible invertible solution of the quadratic equation:

$$\frac{(1+\theta)^2}{(1+\theta)^2\left[\phi_1(1-\phi_2)+2\phi_2\right]+\theta\phi(1)^2} = \frac{r(\sigma_\eta/\sigma_\kappa)}{1+r(\sigma_\eta/\sigma_\kappa)(1+\phi_1+\phi_2)}. \tag{11}$$

If this is possible, then, the remaining parameters are obtained from:

$$\sigma^2_{\kappa^*} = -r\sigma_\eta\sigma_\kappa\frac{\phi(1)^2}{(1+\theta)^2}; \qquad \sigma^2_{\eta^*} = \sigma^2_\eta - \frac{(1+\theta)^2}{\phi(1)^2}\sigma^2_{\kappa^*}.$$

These results make clear that the equivalence is admissible only for negative values of r. When $r = 0$ the solution $\theta = -1$ arises for any value of the ratio $\sigma_\eta/\sigma_\kappa$, in which case the hysteresis parameter is zero and the model can be orthogonally decomposed into a RW trend and a purely AR(2) cycle. No admissible solutions exists for a positive r and in general an UC trend-cycle decomposition with positively correlated disturbances cannot be isomorphic to a cyclical growth model or a model with hysteresis effects. This is so since model (9) implies a spectral density for Δy_t that has a local, but not a global, minimum at the zero frequency.

6.3 Permanent-Transitory Decomposition

A negative r is often interpreted in terms of $\eta_t \rightarrow \kappa_t$; e.g. positive trend disturbances induce negative cyclical shocks. Of course, we could invert the direction of the causality, as in statistics correlation does not necessarily imply causation. Yet another interpretation can be derived using the orthogonalisation

$$\kappa_t = \kappa^*_t + \omega\eta_t, \quad \omega = r\frac{\sigma_\kappa}{\sigma_\eta}, \quad \mathrm{E}(\kappa^*_t, \eta_t) = 0;$$

replacing into (1), with $\theta = 0$, and rearranging, we achieve the following orthogonal decomposition of y_t into a permanent component, $y_t^{(\mathsf{P})}$, and a transitory component, $y_t^{(\mathsf{T})}$:

$$\begin{array}{rcll} y_t & = & y_t^{(\mathsf{P})} + y_t^{(\mathsf{T})}, & t = 1, 2, \ldots, T, \\ \phi(L)\Delta y_t^{(\mathsf{P})} & = & b + [\phi(L) + \omega\Delta]\eta_t & \eta_t \sim \mathrm{NID}(0, \sigma^2_\eta) \\ \phi(L) y_t^{(\mathsf{T})} & = & \kappa^*_t, & \kappa^*_t \sim \mathrm{NID}(0, \sigma^2_{\kappa^*}) \end{array} \tag{12}$$

with $\sigma^2_{\kappa^*} = \sigma^2_\kappa(1-\omega^2)$, $b = \phi(1)\beta$. The permanent component is generated by an ARIMA(2,1,2) process, since the term in square brackets on the right hand side is an MA(2) polynomial. MNZ seem to refer to this decomposition when they speak of nominal shocks that do not affect the trend (κ^*_t) and of a new economy shock that induces a negative output gap: the latter can be associated to the transitory effects of η_t, that amount to $\omega\eta_t/\phi(L)$ (notice that $r < 0$ implies $\omega < 0$). The term "permanent-transitory" decomposition arises by analogy with the Blanchard and Quah (1981) decomposition.

6.4 Illustrative Examples (cont.)

The parameter estimates reported in table 1 rule out the equivalence of the estimated UC($r, 0$) model with UC(0,θ): for both EA and Italy 8 has no admissible solution. As a matter of fact, the spectral density estimated by the former is not a global minimum at the zero frequency.

As far as the equivalence with the cyclical growth - hysteresis model is concerned, for the Euro Area GDP equation (11) has complex roots. However, for the Italian GDP case a real invertible solution is admissible as $\theta = -0.41$; the remaining implied parameter values are $\sigma^2_{\eta*} = 0.3260$ and $\sigma^2_{\kappa*} = 0.0869$; These values are fully coincident with those estimated by maximum likelihood. Hence the Italian GDP provides a case in which the cyclical growth model and the trend cycle decomposition with correlated disturbances provide exactly the same inferences, that are in turn coincident with those arising for the unrestricted ARIMA(2,1,2) model. As a result, alternative explanations of the nature of macroeconomic fluctuations arise with exactly the same likelihood. Finally, the option of deriving a permanent-transitory decomposition from UC(r,0) is open for both series, yielding yet another interpretation.

7 State Space Representation and the Estimation of Unobserved Components

In this section we discuss several facts concerning the real time and smoothed estimates of trends and cycles arising from the various models considered in the previous sections. To accomplish this, we need first to review the state space representation, and the associated algorithms for filtering and smoothing.

7.1 State Space representation

The UC models considered so far admit the time-invariant state space representation:

$$\begin{aligned} y_t &= \boldsymbol{z}'\boldsymbol{\alpha}_t, \qquad t = 1, 2, \dots, T, \\ \boldsymbol{\alpha}_t &= \boldsymbol{T}\boldsymbol{\alpha}_{t-1} + \boldsymbol{c} + \boldsymbol{R}\boldsymbol{\epsilon}_t, \end{aligned} \tag{13}$$

with $\boldsymbol{\epsilon}_t \sim \text{NID}(\mathbf{0}, \boldsymbol{Q})$ and $\boldsymbol{\alpha}_0 \sim \text{NID}(\tilde{\boldsymbol{\alpha}}_0, \boldsymbol{P}_0)$, independently of $\boldsymbol{\epsilon}_t, \forall t$. The treatment of initial conditions is discussed in section 7.2. The state vector has three elements, $\boldsymbol{\alpha}_t = [\mu_t, \psi_t, \psi_t^*]$, and the drift β is considered as a constant effect. Alternatively, we may include β in the state vector using the transition equations for the trend $\mu_t = \mu_{t-1} + \beta_{t-1} + \eta_t$, $\beta_t = \beta_{t-1}$.

For instance, the system matrices for the UC(r, 0) model are:

$$\boldsymbol{z} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \phi_1 & 1 \\ 0 & \phi_2 & 0 \end{bmatrix}, \quad \boldsymbol{c} = \begin{bmatrix} \beta \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{Q} = \begin{bmatrix} \sigma^2_\eta & \sigma_{\eta\kappa} \\ \sigma_{\eta\kappa} & \sigma^2_\kappa \end{bmatrix}$$

For the Beveridge-Nelson decomposition, considered *as a model*, the system matrices are the same except for $\boldsymbol{R}$ and $\boldsymbol{Q}$, which are 3×1 and scalar, respectively:

$$\boldsymbol{R} = \begin{bmatrix} \varrho \\ 1 - \varrho \\ -(\theta_2 + \phi_2 \varrho) \end{bmatrix}, \quad \boldsymbol{Q} = \sigma^2,$$

where $\varrho = \theta(1)/\phi(1)$ is the persistence parameter. On the other hand, for UC(0,θ) we need to replace $\boldsymbol{R}$ and $\boldsymbol{Q}$ by:

$$\boldsymbol{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \theta \end{bmatrix}, \quad \boldsymbol{Q} = \begin{bmatrix} \sigma^2_{\eta^*} & 0 \\ 0 & \sigma^2_{\kappa^*} \end{bmatrix};$$

finally, the state space representation for the cyclical growth model is obtained also replacing $\boldsymbol{z}$ and $\boldsymbol{T}$ by:

$$\boldsymbol{z} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{T} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & \phi_1 & 1 \\ 0 & \phi_2 & 0 \end{bmatrix}.$$

The state space model (13) is in *contemporaneous form*. The *future form*, that is sometimes used to specify the model, differs for the timing of the transition equation, which is written $\boldsymbol{\alpha}_{t+1} = \boldsymbol{T}\boldsymbol{\alpha}_t + \boldsymbol{c} + \boldsymbol{R}\boldsymbol{\epsilon}_t$. Due to the absence of measurement noise and time invariance of the system matrices, if the measurement equation is unaltered, that is $y_t = \boldsymbol{z}'\boldsymbol{\alpha}_t$, the two representations differ only for the (arbitrary) timing of the disturbances. A slightly different representation is obtained if the maintained model is in contemporaneous form and we express it to the future form: this can be done by replacing $\boldsymbol{\alpha}_t$ in the measurement equation with the right hand side of the transition equation and redefining $\boldsymbol{\alpha}^*_t = \boldsymbol{\alpha}_{t-1}$, so as to write:

$$\begin{array}{rcl} y_t & = & \boldsymbol{z}'\boldsymbol{\alpha}^*_t + \boldsymbol{z}'\boldsymbol{R}\boldsymbol{\epsilon}_t, \qquad t = 1, 2, \ldots, T, \\ \boldsymbol{\alpha}^*_{t+1} & = & \boldsymbol{T}\boldsymbol{\alpha}^*_t + \boldsymbol{c} + \boldsymbol{R}\boldsymbol{\epsilon}_t. \end{array} \tag{14}$$

It should be noticed the appearance of measurement noise in the first equation that is correlated with the transition noise. There is nothing "structural" about this component, which appears as a consequence of the operation of forcing the contemporaneous representation into the future form, according to which a "shock" at time t affects the components at time $t + 1$.

7.2 Initialisation

Initialisation deals with the specification of the mean and the covariance matrix of the initial state vector, $\boldsymbol{\alpha}_0$. If we assume that the process $\boldsymbol{\alpha}_t$ has applied since time immemorial ($t \to -\infty$), and if we partition $\boldsymbol{T} = \text{diag}(1, \boldsymbol{T}_\psi)$ and $\boldsymbol{R} = [\boldsymbol{R}'_\mu, \boldsymbol{R}'_\psi]'$, then $\tilde{\boldsymbol{a}}_0 = \mathbf{0}$ and

$$\boldsymbol{P}_0 = \boldsymbol{e}_1\boldsymbol{e}_1'\delta + \begin{bmatrix} 0 & \boldsymbol{d} \\ \boldsymbol{d}' & \boldsymbol{M} \end{bmatrix}$$

where $\delta \to \infty$, $\boldsymbol{e}_1' = [1, 0, 0]$, $\boldsymbol{d} = \boldsymbol{R}_\mu\boldsymbol{Q}\boldsymbol{R}'_\psi(\boldsymbol{I} - \boldsymbol{T}'_\psi)^{-1}$, and $\boldsymbol{M}$ solves the matrix equation $\boldsymbol{M} = \boldsymbol{T}_\psi\boldsymbol{M}\boldsymbol{T}'_\psi + \boldsymbol{R}_\psi\boldsymbol{Q}\boldsymbol{R}'_\psi$. UC models with uncorrelated components have $\boldsymbol{d} = \mathbf{0}'$, whereas, for UC($r$,0), $\boldsymbol{d} = r\sigma_\eta\sigma_\kappa\phi(1)^{-1} \cdot [1, \quad \phi_2]$. However, working out the exact initial KF by letting $\delta \to \infty$, as in (Koopman, 1997) and Durbin and Koopman (2001), it can be checked that the elements of $\boldsymbol{d}$ are wiped away by the limiting operations and thus play no role for inferences. The same result is obtained if one uses the augmentation approach, and in particular the theory in De Jong and Chu-Chun-Lin (1994).

7.3 Kalman Filter

The Kalman filter (Anderson and Moore, 1979), is the well-known recursive algorithm for computing the minimum mean square estimator of $\boldsymbol{\alpha}_t$ and its mean square error (MSE) matrix conditional on $Y_{t-1} = \{y_1, y_2, \ldots, y_{t-1}\}$. Defining $\tilde{\boldsymbol{\alpha}}_{t|t-1} = \text{E}(\boldsymbol{\alpha}_t|Y_{t-1})$, $\boldsymbol{P}_{t|t-1} = \text{E}[(\boldsymbol{\alpha}_t - \tilde{\boldsymbol{\alpha}}_{t|t-1})(\boldsymbol{\alpha}_t - \tilde{\boldsymbol{\alpha}}_{t|t-1})'|Y_{t-1}]$, it is given by the set of recursions:

$$\begin{array}{rclrcl} \xi_t & = & y_t - \boldsymbol{z}'\tilde{\boldsymbol{\alpha}}_{t|t-1}, & f_t & = & \boldsymbol{z}'\boldsymbol{P}_{t|t-1}\boldsymbol{z} \\ & & & \boldsymbol{k}_t & = & \boldsymbol{T}\boldsymbol{P}_{t|t-1}\boldsymbol{z}f_t^{-1} \\ \tilde{\boldsymbol{\alpha}}_{t+1|t} & = & \boldsymbol{T}\tilde{\boldsymbol{\alpha}}_{t|t-1} + \boldsymbol{c} + \boldsymbol{k}_t\xi_t, & \boldsymbol{P}_{t+1|t} & = & \boldsymbol{T}\boldsymbol{P}_{t|t-1}\boldsymbol{T}' + \boldsymbol{R}\boldsymbol{Q}\boldsymbol{R}' - \boldsymbol{k}_t\boldsymbol{k}_t'f_t \end{array} \tag{15}$$

$\xi_t = y_t - \text{E}(y_t|Y_{t-1})$ are the filter innovations or one-step-ahead prediction errors, with variance f_t.

Steady State The innovations and the state one-step-ahead prediction error, $\boldsymbol{x}_t = \boldsymbol{\alpha}_t - \tilde{\boldsymbol{\alpha}}_{t|t-1}$, can be written as

$$\xi_t = \boldsymbol{z}'\boldsymbol{x}_t, \quad \boldsymbol{x}_{t+1} = \boldsymbol{L}_t\boldsymbol{x}_t + \boldsymbol{R}\boldsymbol{\epsilon}_{t+1}, \tag{16}$$

where $\boldsymbol{L}_t = \boldsymbol{T} - \boldsymbol{k}_t\boldsymbol{z}'$. Thus, $\boldsymbol{x}_t$ follows a VAR(1) process that is (asymptotically) stationary if the autoregressive matrix $\boldsymbol{L}_t$, known as the *closed loop matrix* in system theory, converges to a matrix $\boldsymbol{L} = \boldsymbol{T} - \boldsymbol{k}\boldsymbol{z}'$, whose eigenvalues lie all inside the unit circle.

The basic properties that ensure convergence to such stabilising solution are *detectability* and *stabilisability* (see Burridge and Wallis, 1988). For the trend-cycle decompositions considered in this paper they are met if $\phi(L)$ does not display explosive roots or unit roots at the zero frequency. The two conditions imply that, independently of initial conditions, $\boldsymbol{P}_{t+1|t}$ converges at an exponential rate to a steady state solution $\boldsymbol{P}$, satisfying the Riccati equation $\boldsymbol{P} = \boldsymbol{T}\boldsymbol{P}\boldsymbol{T}' + \boldsymbol{R}\boldsymbol{Q}\boldsymbol{R}' - \boldsymbol{k}\boldsymbol{k}'f$, with $\boldsymbol{k} = \boldsymbol{T}\boldsymbol{P}\boldsymbol{z}f^{-1}$ and $f = \boldsymbol{z}'\boldsymbol{P}\boldsymbol{z}$, and the Kalman gain vector $\boldsymbol{k}$ is such that $\boldsymbol{L}$ has all its eigenvalues inside the unit circle.

7.4 Real time estimates

The real time or concurrent estimates of the states and the estimation error covariance matrix are given respectively by:

$$\tilde{\boldsymbol{\alpha}}_{t|t} = \tilde{\boldsymbol{\alpha}}_{t|t-1} + \boldsymbol{P}_{t|t-1}\boldsymbol{z}f_t^{-1}\xi_t, \quad \boldsymbol{P}_{t|t} = \boldsymbol{P}_{t|t-1} - \boldsymbol{P}_{t|t-1}\boldsymbol{z}\boldsymbol{z}'\boldsymbol{P}_{t|t-1}f_t^{-1}. \tag{17}$$

The estimated unobserved components in $\tilde{\boldsymbol{\alpha}}_{t|t}$ are the same as those arising from the BN decomposition of the implied ARIMA reduced form representation. The MA parameters of the reduced form representation can be uniquely derived from the steady state using $\boldsymbol{P}\boldsymbol{z}f^{-1}$, whose first element is the persistence parameter. Notice, however, that in the steady state we need $\boldsymbol{z}\boldsymbol{z}'f^{-1}$ to be equal to the pseudo-inverse of $\boldsymbol{P}$ for the components to be estimated with zero error, i.e. observable with respect to current and past information. For the BN model $f = \sigma^2 = \boldsymbol{Q}$, $\boldsymbol{P}\boldsymbol{z}f^{-1} = \boldsymbol{R}$ and $\boldsymbol{P} = \boldsymbol{R}\boldsymbol{Q}\boldsymbol{R}'$, $\boldsymbol{k} = \boldsymbol{T}\boldsymbol{R}$, which ensures that when the system has reached a steady state, the components are estimated in real time with zero mean square error.

7.5 Single Source of Error and Innovation State Space Models

The BN decomposition can be viewed as the "structural" representation of *Single Source of Error* (SSE, Snyder, 1985, Hyndman *et al.*, 2002) and *Steady State Innovation Models* (SSIM, Casals, Jerez and Sotoca, 2002). In this section we establish the connection among these alternative representations of the same underlying model.

Recalling the state space representation of the BN decomposition, which is (13) with scalar $\boldsymbol{\epsilon}_t = \xi_t$ and $\boldsymbol{R} = [\varrho, (1-\varrho), -(\theta_2+\phi_2\varrho)]$, we use manipulations similar to those which led to (14) in two steps: we first replace $\boldsymbol{\alpha}_t$ in the measurement equation by the right hand side of the transition equation to obtain the single source of error representation:

$$y_t = \boldsymbol{z}'\boldsymbol{T}\boldsymbol{\alpha}_{t-1} + \xi_t, \;\; \boldsymbol{\alpha}_t = \boldsymbol{T}\boldsymbol{\alpha}_{t-1} + \boldsymbol{c} + \boldsymbol{R}\xi_t,$$

the new measurement equation features ξ_t since $\boldsymbol{z}'\boldsymbol{R} = \boldsymbol{z}'\boldsymbol{r} = 1$.

Next, we posit $\boldsymbol{\alpha}_t^* = \boldsymbol{T}\boldsymbol{\alpha}_{t-1}$ and on premultiplying both sides of the transition equation by $\boldsymbol{T}$, we write:

$$\begin{array}{rcl} y_t &=& \boldsymbol{z}'\boldsymbol{\alpha}_t^* + \xi_t, \qquad t = 1, 2, \ldots, T, \\ \boldsymbol{\alpha}_{t+1}^* &=& \boldsymbol{T}\boldsymbol{\alpha}_t^* + \boldsymbol{c} + \boldsymbol{k}\xi_t, \end{array} \tag{18}$$

with $\boldsymbol{k} = \boldsymbol{T}\boldsymbol{R}$. This is the innovation form of the model in the steady state, as can be seen by comparing (18) with the first and the last row of the KF equations in (15). The Kalman gain identity is $\boldsymbol{k} = \boldsymbol{T}\boldsymbol{R}$.

We stress that both forms are available for any UC model, and in fact the SSIM arises from the filtering operation, and the SSE from the updating equations, but only for the BN model a one to one correspondence holds. If SSIM or SSE are estimated as a model there is no way of recovering information on multiple source of errors.

The emphasis on the "exact" nature of inference for SSE and SSIM models is misplaced: the property that components are estimated in real time, state Casals, Jerez and Sotoca (2002) in their concluding remarks (p. 563),

> "ensures coherence between the properties of the theoretical and empirical components, provides a rigorous statistical foundation for using the empirical components as observable and mutually independent time series, and guarantees that these components will not change as the sample increases".

First and foremost, the BN illustration shows that both SSIM and SSE representations feature (perfectly correlated) measurement noise that is not present in the BN model, and thus are not *coherent* in this respect with the maintained model. As a matter of fact, Casals, Jerez and Sotoca interpret the one-step-ahead forecast error as an estimate of the irregular component. Moreover, the timing of the disturbances is not the same as the original model, when the latter is expressed in contemporaneous form. Secondly, the empirical components cannot be mutually independent, being driven by the same disturbance. Finally, their observability in real time, and thus the absence of revision is simply a consequence of the model formulation, i.e. a property and non necessarily an advantage.

We do not need much theory to show that when (18) is interpreted as a model, and the system is stabilisable and detectable, the states are observed: substituting the expression for ξ_t in the measurement equation into the transition equation, yields $\boldsymbol{\alpha}^*_{t+1} = \boldsymbol{L}\boldsymbol{\alpha}^*_t + \boldsymbol{c} + \boldsymbol{k}y_t$, that is

$$\boldsymbol{\alpha}^*_{t+1} = \boldsymbol{k}y_t + \boldsymbol{L}\boldsymbol{k}y_{t-1} + \boldsymbol{L}^2\boldsymbol{k}y_{t-2} + \cdots.$$

This expression makes it apparent that the states are a linear combination of *past* observations. This is so since the "states" are in fact one-step-ahead predictions.

In conclusions, SSIM and SSE are useful for prediction, but when they are used for estimation of unobserved components they are prone to a number of inconsistencies and a variety of interpretative issues.

7.6 Smoothing and Final Estimates

We can keep track of revisions, due to the accrual of further observations, by using a *fixed-point smoothing* algorithm. Elaborating results in de Jong (1989), and assuming that the system has reached a steady state, we have, for a fixed t and for $l \geq 0$, the following smoothing recursions:

$$\begin{array}{rclrcl} \tilde{\boldsymbol{\alpha}}_{t|t+l} & = & \tilde{\boldsymbol{\alpha}}_{t|t} + \boldsymbol{P}\boldsymbol{L}'\boldsymbol{r}_{t|t+l}, & \boldsymbol{P}_{t|t+l} & = & \bar{\boldsymbol{P}}_{t|t} - \boldsymbol{P}\boldsymbol{L}'\boldsymbol{N}_{t|t+l}\boldsymbol{L}\boldsymbol{P}, \\ \boldsymbol{r}_{j|t+l} & = & \boldsymbol{L}'\boldsymbol{r}_{j+1|t+l} + \boldsymbol{z}f^{-1}\xi_{j+1}, & \boldsymbol{N}_{j|t+l} & = & \boldsymbol{L}'\boldsymbol{N}_{j+1|t+l}\boldsymbol{L} + \boldsymbol{z}\boldsymbol{z}'f^{-1}, \end{array} \tag{19}$$

$j = t+l, t+l-1, \ldots, t$, where $\bar{\boldsymbol{P}}_{t|t} = \boldsymbol{P} - \boldsymbol{P}\boldsymbol{z}\boldsymbol{z}'\boldsymbol{P}f^{-1}$ and the backwards recursions are initialised $\boldsymbol{r}_{t+l|t+l} = \boldsymbol{0}$, $\boldsymbol{N}_{t+l|t+l} = \boldsymbol{0}$.

Now, as $l \to \infty$ (i.e. assuming a doubly infinite sample), $\boldsymbol{r}_{j|t+l}$ is a backward first order stationary vector autoregression, and $\boldsymbol{N}_{j|t+l}$ is its covariance matrix. The final state estimation error covariance matrix, denoted $\boldsymbol{P}_{t|\infty}$, solves $\boldsymbol{P}_{t|\infty} = \boldsymbol{P} - \boldsymbol{P}\boldsymbol{N}\boldsymbol{P}$, where $\boldsymbol{N}$ is the steady state solution of the backward smoothing equation, $\boldsymbol{N}_{j|t+l} = \boldsymbol{L}'\boldsymbol{N}_{j+1|t+l}\boldsymbol{L} + \boldsymbol{z}\boldsymbol{z}'f^{-1}, j = t+l, \ldots, t$, as $l \to \infty$; a unique stable solution for $\boldsymbol{N}$ exists provided the characteristic roots of $\boldsymbol{L}$ are less than unity in modulus, which is already the condition for a steady state solution. The elements of the solution are obtained from

$$\text{vec}(\boldsymbol{N}) = (\boldsymbol{I} - \boldsymbol{L}' \otimes \boldsymbol{L}')^{-1}\text{vec}(\boldsymbol{z}\boldsymbol{z}' f^{-1}).$$

Hence, $\boldsymbol{P}_{t|\infty}$ contains the final estimation error covariance matrix, and can be written:

$$\boldsymbol{P}_{t|\infty} = \bar{\boldsymbol{P}}_{t|t} - \boldsymbol{P}(\boldsymbol{z}\boldsymbol{z}' f^{-1} - \boldsymbol{N})\boldsymbol{P}.$$

The second term on the right hand side, which is obviously positive semi-definite, measures the total reduction in the estimation uncertainty as we go from the real time to the final estimates.

8 Illustrative Examples (cont.)

Figure 3 displays the smoothed estimates of the components of the EA GDP arising from the UC(r, 0) and UC(0,θ) models estimated in section 5. We also present the real time estimates, $\tilde{\psi}_{t|t}$, in the bottom panels along with their 95% confidence interval.

According to the trend estimates for UC(r,0), trend output is above actual output at the beginning of the 70ies; it peaks at 1973.3 and starts declining until it reaches a trough in 1974.2. During this decline we observe a positive and high cycle, as implied by the strong and negative correlation between the two. A similar behaviour is found around 1979 and 1991. On the other hand, the final trend estimates for UC(0,θ) are less "volatile" and less related to the cyclical component; we adopt this terminology since even in a doubly infinite sample the estimates of the components will be correlated - the correlation can be computed from the elements of the matrix $\boldsymbol{P}_{t|\infty}$.

The two models produce very different smoothed cycle estimates: the estimates of the AR parameters in the UC(0,θ) case imply a stationary root at the zero frequency with multiplicity 2, whereas those for UC(r,0) imply a short run cycle with a period of about three years. It is also remarkable the difference between the real time, $\tilde{\psi}_{t|t}$, and final estimates of the cycle, $\tilde{\psi}_{t|T}$, especially for the model with correlated components. When the cycle is estimated in real time (bottom left panel), UC(r,0) lends support to the notion that this component represents a minor source of variation; we recall that the real time estimates cycle $\tilde{\psi}_{t|t}$ arising from UC(r, 0) is coincident with the BN cycle extracted from the unrestricted ARIMA(2,1,2). The latter is characterised by a perfect negative correlation (persistence is greater than 1) and has a non invertible ARMA(2,1) representation; as matter of fact, the parameter values reported in table 1 for EA imply a value for the ϑ^* coefficient in (5) that is equal to -1.02.

However, for UC(r,0) the cycle is estimated in real time with non zero mean square error and the picture changes radically as we proceed to construct the final estimates using also future observations. These contradict the assertion that the cycle has a small amplitude, as it ranges from about -2.4% to +4.0%, as a percentage of GDP. Moreover, the final estimates have a much reduced standard error as compared to the real time ones. In particular, the increase in the reliability of the cycle estimates using a doubly infinite sample is as large as 88%. This quantity is defined as the percentage reduction in the estimation error variance when we compare the real time estimates with the final ones and is computed as: $100[\bar{P}^{(\psi)}_{t|t} - P^{(\psi)}_{t|\infty}]/\bar{P}^{(\psi)}_{t|t}$, where, using results presented in section 7, and in particular (19), $\bar{P}^{(\psi)}_{t|t}$ is the steady state estimation error variance of the real time cyclical component and $P^{(\psi)}_{t|\infty}$ is that of the corresponding final estimates, using a doubly infinite sample.

If the BN decomposition is estimated as a *model*, that is we set up a state space model consisting of equations (3)-(5), after processing a suitable small number of observations the real time and final estimates are fully coincident. On the other hand, the estimates arising from UC(r,0) are subject to large revisions as new observations become available: indeed, for the estimated variance ratio $\sigma^2_\eta/\sigma^2_\kappa$ and AR parameters, a negative r implies that the distribution of the weights for extraction of the cycle, based on a doubly infinite sample, are highly skewed towards the future.

It is remarkable that in the more extreme case, when $r = -1$, the cycle is estimated with zero mean square error using a doubly infinite sample, but the real time estimates are characterised by high uncertainty, and we get a 100% increase in reliability from processing future observations. The model has a single source of disturbances, but it implies a non invertible ARIMA(2,1,2) representation, and thus the latter is not an innovation, but can be written as a linear combination of the current and future values of Δy_t.

On the contrary, when $r = 1$, the single source of disturbances is an innovation in a strict sense; it can be checked that the ACGF identity $g_r(L) = g(L)$ admits the solution $r = 1$, $\sigma_\eta = \sigma\theta(1)/\phi(1)$, $\sigma_\kappa = \sigma[1 - \theta(1)/\phi(1)]$, which implies $\phi_2\theta(1) + \theta_2\phi(1) = 0$. Thus, the real time estimates have the same AR(2) representation as the true component ($\vartheta^* = 0$); the process generating them is coincident with the maintained model for the unobserved component, so that current and past (i.e. real time) information is all we need to form this estimate.

In conclusion, if we accept that trend and cyclical disturbances are negatively correlated, then we must be willing to accept also that essential information for assessing the cyclical pattern lies in future observations and thus that our signals are prone to high revisions.

Figure 4 presents the estimated components of Italian GDP. The overall comments are unchanged except for the fact that the cycles extracted by UC(r,0) and UC(0,θ) have in this case about the same periodicity. We also present two alternative characterisations of macroeconomic fluctuations arising from the cyclical growth model, which is observationally equivalent to UC(r,0) and to the unrestricted ARIMA model. The first is the deviation cycle extracted under the hysteresis hypothesis (see model (10)); the latter modifies the trend permanently since it is integrated in the trend with a weight of 0.59. The second is the smoothed cycle in Δy_t based on (9). We can only resort to our prior to attach a preference to these alternative representations.

9 The Role of Seasonal Adjustment

The analysis of macroeconomic fluctuations usually relies on quarterly seasonally adjusted series. This raises the obvious issue as to whether seasonal adjustment can be considered as a neutral operation, in the sense that it does not alter the main stylised facts. The presence of a correlation between trend and cycle disturbances is one of those facts, given the relevance that the literature attaches to it.

To investigate this issue we perform a very simple Monte Carlo experiment, by which 1000 series of length $T = 140$ are generated according to $y_t = \mu_t + \psi_t + \gamma_t$, with independent trends and cycles, represented as UC(0,0) in (1); γ_t is a quarterly seasonal component, with trigonometric representation: $\gamma_t = \gamma_{1t} + \gamma_{2t}$, resulting from the sum of an annual non stationary cycle $(1 + L^2)\gamma_{1t} = \varpi_{1t}$ and a biannual one, $(1 + L)\gamma_{2t} = \varpi_{2t}$, with $\varpi_{1t} \sim \mathrm{NID}(0, \sigma^2_\omega)$ and $\varpi_{2t} \sim \mathrm{NID}(0, 0.5\sigma^2_\omega)$, independently of each other and of η_t and κ_t.

The cycle autoregressive parameters are written as $\phi_1 = -2\cos\lambda_c, \phi_2 = \rho^2$, where $\rho = 0.9$ and λ_c can take the two values $2\pi/12$ and $2\pi/32$ corresponding to a period of 3 (12 quarters) and 8 years (32 quarters), respectively. The trend-seasonal signal ratio is always kept at $\sigma^2_\eta/\sigma^2_\omega = 20$, whereas for $\sigma^2_\eta/\sigma^2_\kappa$ we consider three values, Low: $\sigma^2_\eta/\sigma^2_\kappa = 1/3$; Medium: $\sigma^2_\eta/\sigma^2_\kappa = 3$; High: $\sigma^2_\eta/\sigma^2_\kappa = 30$. The combination of these values with the two cycle periods gives 6 data generating processes in total.

For each simulation we fit the true model model and construct a seasonally adjusted (SA) series by removing from the simulated series the smoothed estimates of the seasonal component; the UC(r,0) is the fitted to the series. In the presentation of the results we label this experiment as SA-UC(r,0). Moreover, to characterise the small sample distribution of the correlation coefficient when the true value is $r = 0$, we estimate model a trend plus cycle plus seasonal model with correlated trend and cycle disturbances, that is UC(r, 0) plus an orthogonal seasonal component. We shall refer to this experiment with TCS(r, 0).

Figure 5 plots the distribution of the estimated r for SA-UC(r,0) and TCS(r,0) in the six cases. The histograms clearly point out that seasonal adjustment biases the estimates of the correlation coefficient, increasing the the evidence for a negative correlation. In general, the problem is lessened as we move away from the fundamental seasonal frequency (a yearly cycle), as the histograms for the 32 quarters cycle suggest.

Also, the panels in the second and the fourth columns highlight that the small sample distribution of r estimated on the unadjusted data is highly nonstandard, suffering from the same pile-up problem at ± 1 that was observed for the bootstrap distribution in section 5. Experimentation suggests that we need a much larger sample size to have r distributed symmetrically around its true zero value.

10 The Role of Temporal Aggregation

Violation of the conditions under which a series admits an orthogonal trend-cycle decomposition may well be the consequence of temporally aggregating a flow variable. On the other hand, the only way in which systematic can affect the decomposability of the model into orthogonal components is via the small sample properties of the parameter estimates.

In this section we assume that observations are available on an aggregate series $Y_n, n = 1, 2, \ldots, N$, that is obtained either by systematically sampling a stock variable or by aggregating a flow. Let s denote the aggregation period; if we denote the disaggregated series by $y_t, t = 1, 2, \dots, Ns$, in the former case $Y_n = y_{sn}$; the latter can be viewed as a systematic sample of $Y_n = \sum_{j=0}^{s-1} y_{sn-j} = S(L)y_t,\ S(L) = 1 + L + \cdots + L^{s-1}$, where the sample is taken at times $t = ns$;

Let us consider systematic sampling first: since $Y_n - Y_{n-1}$ is a systematic sample of $\Delta_s y_t$, its SGF is related to that of Δy_t, $g_{\Delta y}(\lambda)$, via the expression (Harvey, sec. 6.3.5):

$$g_{SS}(\lambda) = \frac{1}{s}\sum_{j=0}^{s-1} |S(e^{-\imath\omega_j})|^2 g_{\Delta y}(\omega_j), \tag{20}$$

where $\omega_j = s^{-1}(\lambda + 2\pi j)$, and

$$|S(e^{-\imath\omega_j})|^2 = S(e^{-\imath\omega_j})S(e^{\imath\omega_j}) = \begin{cases} \frac{1-\cos\omega}{1-\cos\omega_j}, & \omega_j \neq 0 \\ s, & \omega_j = 0 \end{cases}$$

is the power transfer function of the filter $S(L)$ evaluated at ω_j. Result (20) follows from application of the well known *folding* formula to the process $S(L)y_t$.

If the disaggregated series follows an UC(0,θ) model, so that the SGF of Δy_t is $g_{\Delta y}(\lambda) = \sigma^2_{\eta*} + 2(1-\cos\lambda)g_\psi(\lambda)$, where $g_\psi(\lambda)$ is the SGF of the cyclical component, then, using $\sum_{j=0}^{s-1} |S(e^{-\imath\omega_j})|^2 = s^2$, (20) specialises as

$$g_{SS}(\lambda) = s\left[\sigma^2_{\eta*} + 2(1-\cos\lambda)\frac{1}{s}\sum_{j=0}^{s-1} g_\psi(\omega_j)\right] \tag{21}$$

and the aggregated series can still be decomposed into orthogonal trend and cycle, with SGF $s^{-1}\sum_{j=0}^{s-1} g_\psi(\omega_j)$.

In the case of temporal aggregation, $Y_n - Y_{n-1}$ is a systematic sample of $\Delta_s S(L)y_t$ and thus the SGF of the aggregated series, $g_{TA}(\lambda)$, will be related to that of the disaggregated process as follows:

$$g_{TA}(\lambda) = \frac{1}{s}\sum_{j=0}^{s-1} |S(e^{-\imath\omega_j})|^4 g_{\Delta y}(\omega_j) \tag{22}$$

When the disaggregated series is an UC(0,θ) process, (22) becomes:

$$g_{TA}(\lambda) = s^{-1}\left\{s^4\sigma^2_{\eta*} + (1-\cos\lambda)^2\sum_{j=0}^{s-1}(1-\cos\omega_j)^{-2}\left[\sigma^2_\eta + 2(1-\cos\omega_j)g_\psi(\omega_j)\right]\right\} \tag{23}$$

Expression (23) has a complicated form and the decomposability into orthogonal components will arise only under very special conditions. The reduced form will be, in general, ARIMA(2,1,3) and will be decomposable into a RW trend plus ARMA(2,1) cycle plus irregular with correlated disturbances or into a RW trend plus ARMA(2,2) cycle with correlated components. Since temporal aggregation is a linear operation, another option is to decompose Y_n into an orthogonal IMA(1,1) trend and ARMA(2,2) cycle.

The ambiguities that aggregation of flow variables creates can be seen by working out the state space representation of Y_n (see Harvey, 1989). When the disaggregated model is (13), the state space model for the aggregate is (see Harvey, 1989):

$$\begin{array}{rcl} Y_n & = & \boldsymbol{z}'(\sum_{j=1}^{s}\boldsymbol{T}^j)\boldsymbol{\alpha}_{n-1} + \boldsymbol{z}'\left[\sum_{j=0}^{s-1}\sum_{i=0}^{j}\boldsymbol{T}^i\right]\boldsymbol{c} + u_t, \qquad n = 1, 2, \ldots, N, \\ \boldsymbol{\alpha}_n & = & \boldsymbol{T}^s\boldsymbol{\alpha}_{n-1} + (\sum_{j=0}^{s-1}\boldsymbol{T}^j)\boldsymbol{c} + \tilde{\boldsymbol{\epsilon}}_n, \end{array} \tag{24}$$

where

$$u_t = \boldsymbol{z}'\sum_{j=0}^{s-1}(\sum_{i=j}^{s-1}\boldsymbol{T}^{i-j})\boldsymbol{R}\boldsymbol{\epsilon}_{ns-j}, \quad \tilde{\boldsymbol{\epsilon}}_n = \sum_{j=0}^{s-1}\boldsymbol{T}^j\boldsymbol{R}\boldsymbol{\epsilon}_{ns-j}.$$

The representation (24) is already in the future state space form as can be seen on defining $\boldsymbol{\alpha}_n^* = \boldsymbol{\alpha}_{n-1}$. The new feature is the presence of the disturbance u_t in the measurement equation, that is correlated with the state disturbances. Interpreting u_t as correlated measurement noise is possible, but arbitrary. Other options, such as incorporating u_t into one of the components are arbitrary as well. Also, the BN decomposition will always estimate a RW trend, but the temporal aggregation of the trend component will give and IMA(1,1) process. These indeterminacies are resolved if the model is specified at the disaggregated frequency and estimated on the available series.

This leads us to the main point of this section. Suppose that the underlying model is UC(0,0) at the monthly level, but observations are available on the quarterly aggregate ($s = 3$). Can temporal aggregation explain the stylised fact that the spectral density of $Y_n - Y_{n-1}$ is not a minimum at the zero frequency? For this purpose we consider the Italian GDP and we fit (23) to the sample periodogram by maximum likelihood, as illustrated in section 5.

The maximised likelihood is -140.38 and the parameter estimates are $\hat{\sigma}^2_{\eta*} = 0.0246$, $\hat{\sigma}^2_{\kappa*} = 0.0001$, $\hat{\phi}_1 = 1.94$, $\hat{\phi}_2 = -0.97$, implying a period of about 3 years (11 quarters), which amounts to the same period estimated by the quarterly UC(r,0). The model provides a good fit $Q(12) = 7.37$ normality 2.78, and yields a slightly greater likelihood than UC(r,0) (compare table 1). Figure 6 shows how the model fits the raw periodogram; for comparison we report the parametric spectral density of UC(r,0). We notice that the spectral density estimate at zero and the implied persistence is the same, but the temporally aggregate UC(0,0) model (referred to as TA-UC(0,0)) has a sharper peak at the cyclical frequency. The estimated cycle, resulting from the aggregation of the monthly cycle, is plotted in the bottom panel and it is, roughly speaking, a compromise between those estimated by the quarterly UC(r,0) and by UC(0,θ).

The empirical findings in Rossana and Seater (1995) illustrate that temporally aggregated flows systematically show higher long run persistence with respect to the underlying disaggregated data. These results are not only a reflection of the small sample properties of the estimates, but also a theoretical consequence of temporal aggregation of flows. Persistence is the square root of the ratio $g(0)/\sigma^2$, where

$$\sigma^2 = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\lambda)d\lambda\right)$$

is the prediction error variance (p.e.v.). Now, while it is straightforward to establish $g_{SS}(0) = sg_{\Delta y}(0)$ and $g_{TA}(0) = s^3 g_{\Delta y}(0)$, it proves difficult to derive analytical results on the effects of aggregation on persistence. However, we can easily prove that persistence of an aggregated flow is no smaller than that of a systematic sample. Applying the Cauchy-Schwartz inequality to the p.e.v. of an aggregated flow, it follows that $\sigma^2_{TA} \leq s^2\sigma^2_{SS}$, moreover, since $g_{TA}(0) = s^{-2}g_{SS}(0)$, we establish the result: $g_{TA}(0)/\sigma^2_{TA} \geq g_{SS}(0)/\sigma^2_{SS}$.

In conclusion, temporal aggregation of flow variables is non neutral with respect to the main stylised facts concerning macroeconomic fluctuations, such as persistence and correlated disturbances.

11 Multivariate Analysis

The previous analyses have been typically univariate. The issue that needs to be addressed at this stage is whether bringing in more information about the nature of economic fluctuations using related series can cast some light on correlated disturbances.

We set off reviewing some previous empirical results. Clark (1989) estimated a bivariate model of U.S. real output and unemployment grounded on the relationship between cyclical movements in output and unemployment known as Okun's law. The model for output is UC(r,0), and the unemployment rate is decomposed into a (driftless) random walk trend, unrelated to that in output, and a cyclical component that is a linear combination of the current and past value of the cycle in output. Clark estimated the correlation coefficient r to be equal to a nonsignificant -.12, with asymptotic 90% confidence interval (-0.4,0.3).

Jäger and Parkinson (1994) estimated bivariate UC models of real GDP and unemployment to examine the presence of hysteresis, according to which cyclical unemployment has an effect on the natural (trend) rate. They find that hysteresis effects are negligible in explaning the dynamics of U.S. unemployment, but are substantial for the Canadian, German and the U.K. unemployment rates.

Proietti, Musso and Westermann (2002) estimated a multivariate model made up of five time series equations for the Euro area Solow's residual, the labour force participation rate, the unemployment rate, capacity utilisation and the consumer price index, implementing the production function approach, augmented by a triangle model of inflation (see Gordon, 1997), to the measurement of potential output and the output gap for the Euro Area. They entertain hysteresis models and find mixed evidence for the unemployment series. They, however, prefer a specification featuring pseudo-integrated cycles that is at least as effective in explaining the persistence of the labour market variables.

We now proceed to a bivariate illustration concerning the Euro area GDP and consumer prices (p_t, logarithms). Within this framework the cycle in output takes the more specialised notion of an output gap, a measure of inflationary pressures, and the trend is the level of output that is consistent with stable inflation (potential output). The model is made up of the output equation, which is alternatively specified as UC(0,0) and UC(r,0), as given in (1), and the price equation is a structural version of Gordon's triangle model

Table 2: Parameter estimates and diagnostics for bivariate models of quarterly euro area log GDP (y_t) and the logarithm of the consumer price index (p_t), 1970.1-2002.2. Standard errors in parenthesis.

	UC(0,0)	UC(r,0)
σ^2_η ($\sigma^2_{\eta^*}$)	0.2231	0.2546
σ^2_κ ($\sigma^2_{\kappa^*}$)	0.0713	0.0987
ϕ_1	1.67	1.66
s.e	(0.05)	(0.05)
ϕ_2	-0.71	-0.71
s.e	(0.11)	(0.10)
r	0(r)	-0.26
s.e	-	(0.36)
$\sigma^2_{\eta\pi}$	0.0476	0.0468
$\sigma^2_{\zeta\pi}$	0.0000	0.0000
σ^2_ω	0.0000	0.0000
$\theta_{\pi 0}$	0.22	0.20
s.e	(0.04)	(0.05)
$\theta_{\pi 1}$	-0.20	-0.18
s.e	(0.04)	(0.04)
	Diagnostics and goodness of fit	
loglik	-133.28	-133.04
$Q(8)$ y_t	10.70	11.33
$Q(8)$ p_t	6.43	6.99
Normality y_t	9.18	9.51
Normality p_t	5.91	5.34

of inflation, specified as follows:

$$p_t = \tau_t + \gamma_t + \textstyle\sum_k \delta_k x_{kt}$$

$$\begin{aligned} \tau_t &= \tau_{t-1} + \pi^*_{t-1} + \eta_{\pi t} & \eta_{\pi t} \sim \text{NID}(0, \sigma^2_{\eta\pi}), \\ \pi^*_t &= \pi^*_{t-1} + \theta_\pi(L)\psi_t + \zeta_{\pi t} & \zeta_{\pi t} \sim \text{NID}(0, \sigma^2_{\zeta\pi}). \end{aligned}$$

where the regressors are commodity prices x_{kt} and the nominal effective exchange rate of the Euro, a level shift variable for 1974.1, and γ_t is a quarterly seasonal component, see section 9. The only link between the prices and output equations is the presence of ψ_t as a determinant of underlying inflation, π^*_t, where $\theta_\pi(L) = \theta_{\pi 0} + \theta_{\pi 1} L$.

Table 2 reports selected estimation results concerning the model assuming uncorrelated disturbances (first column) and that with correlated ones (second column). The estimated correlation coefficient is -0.26 and the likelihood ratio of the restriction $r = 0$ is not significant. What is more, unlike the univariate case, the estimates of the trend and cycle in output closely agree with those of the model with uncorrelated disturbances. These are displayed in figure 7, along with estimates of underlying inflation, which is that part of observed inflation, devoid of seasonal fluctuations, related to the output gap, which is identified as the component π^*_t in the price equation. The autoregressive parameter estimates imply for both models a period of 10 years. It is also remarkable the reduction in the estimation error variance of the component ψ_t, compared to the univariate case (see fig. 3).

12 Concluding Remarks

The main conclusions of this paper is that the characterisation of macroeconomic fluctuations is by and large an open issue. On the one hand, models with correlated disturbances seem to improve the fit of UC decompositions to macroeconomic time series; this is true at least for the Italian GDP. This finding is in part self-evident, since we entertain a more general model, and needs be interpreted with the following *caveat*: all our results are conditional on a particular ARIMA reduced form, which is itself an additional source of uncertainty in real life. For instance, Harvey and Jäger (1994) entertained an orthogonal trend-cycle decomposition to the U.S. real GDP series, allowing for a stochastic slope in the trend, so that the latter is an $I(2)$ process. Discriminating among UC models unconditionally, i.e. without assuming a particular reduced form, is a far more complex issue, due to the unavailability of a common estimable reduced form.

On the other hand, several additional points were raised and illustrated, that mitigate this finding and are hereby summarised:

- Given the sample sizes typically available for macroeconomic time series the properties of the sampling distribution of the correlation parameter raise great concern. It was shown that asymptotic inferences do not provide a reliable guidance over them.

- Models with correlated components raise several interpretative issues, as under certain conditions they result observationally equivalent to models that provide different and equally plausible explanations of the nature of macroeconomic fluctuations. For the Italian GDP, the cyclical growth model and the hysteresis model provide exactly the same likelihood estimates.

- When we come to investigate the consequences of having highly and negatively correlated disturbances for signal extraction, it turns out that similarity with the BN decomposition does not carry over to the smoothed inferences, the estimated models implying that most information about the components is carried by future observations. Large revisions are thus to be expected.

- Seasonal adjustment and temporal aggregation can significantly affect the findings about correlated disturbances. In particular, the estimates of the correlation between trend and cycle disturbances are biased towards high and negative values.

- Univariate time series analysis cannot be demanded to solve such a controversial issue. Multivariate analysis can help. Our illustration, concerning a bivariate model of output and prices shows that the estimate of the correlation is substantially reduced and that the cycle in output can be estimated with increased reliability.

- The statistical literature has attached much significance to the restrictive nature of models with orthogonal components. However, when single source of errors and innovation representations are considered as a model, several inconsistencies arise. The emphasis on the exact nature of the resulting decompositions and on the absence of revision is misplaced and potentially misleading.

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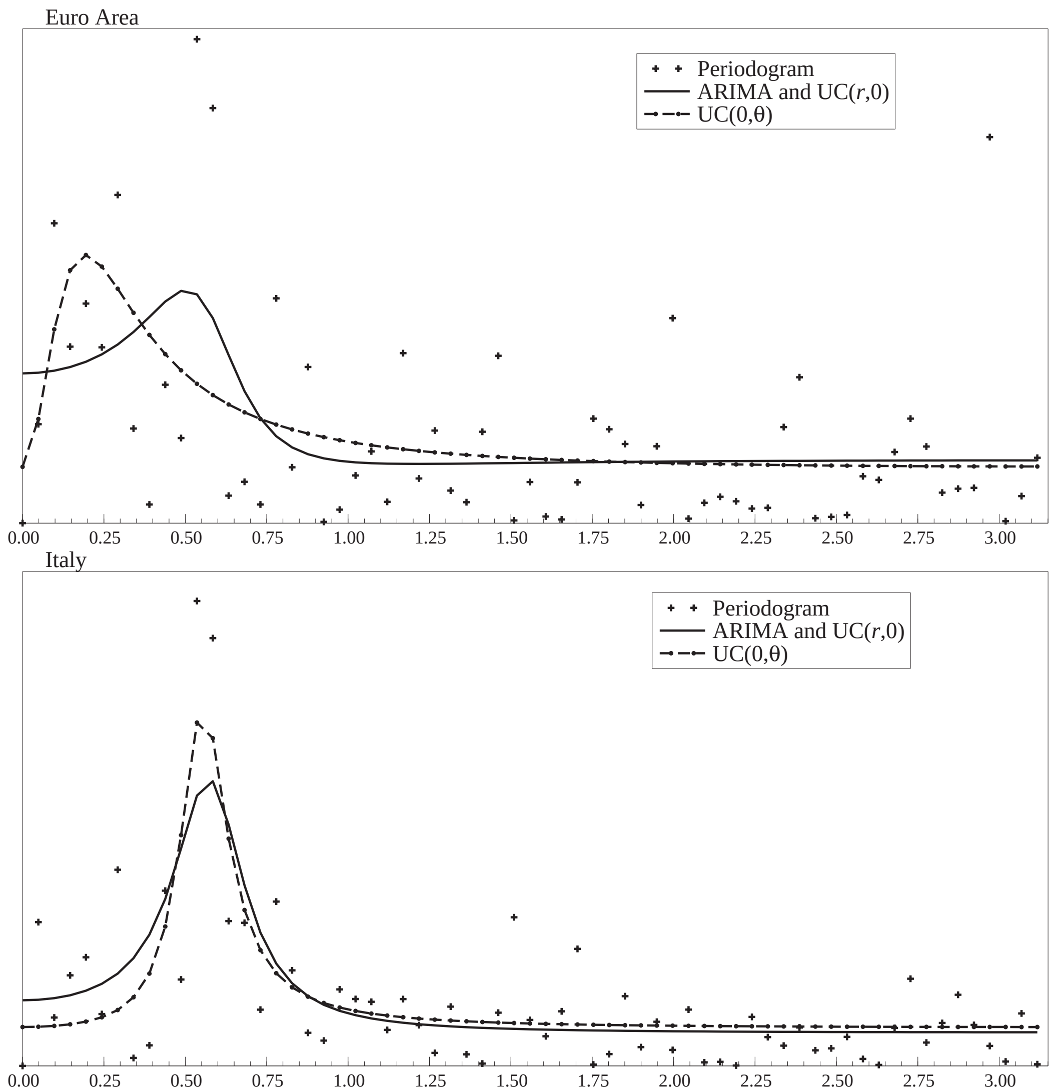

Figure 1: Euro Area and Italian GDP, 1970.1-2002.2. Periodogram, $I(\lambda_j)$, and parametric spectral densities of Δy_t, $g_m(\lambda_j)/(2\pi)$, estimated by the ARIMA(2,1,2) model, the UC(r,0), UC(0,θ).

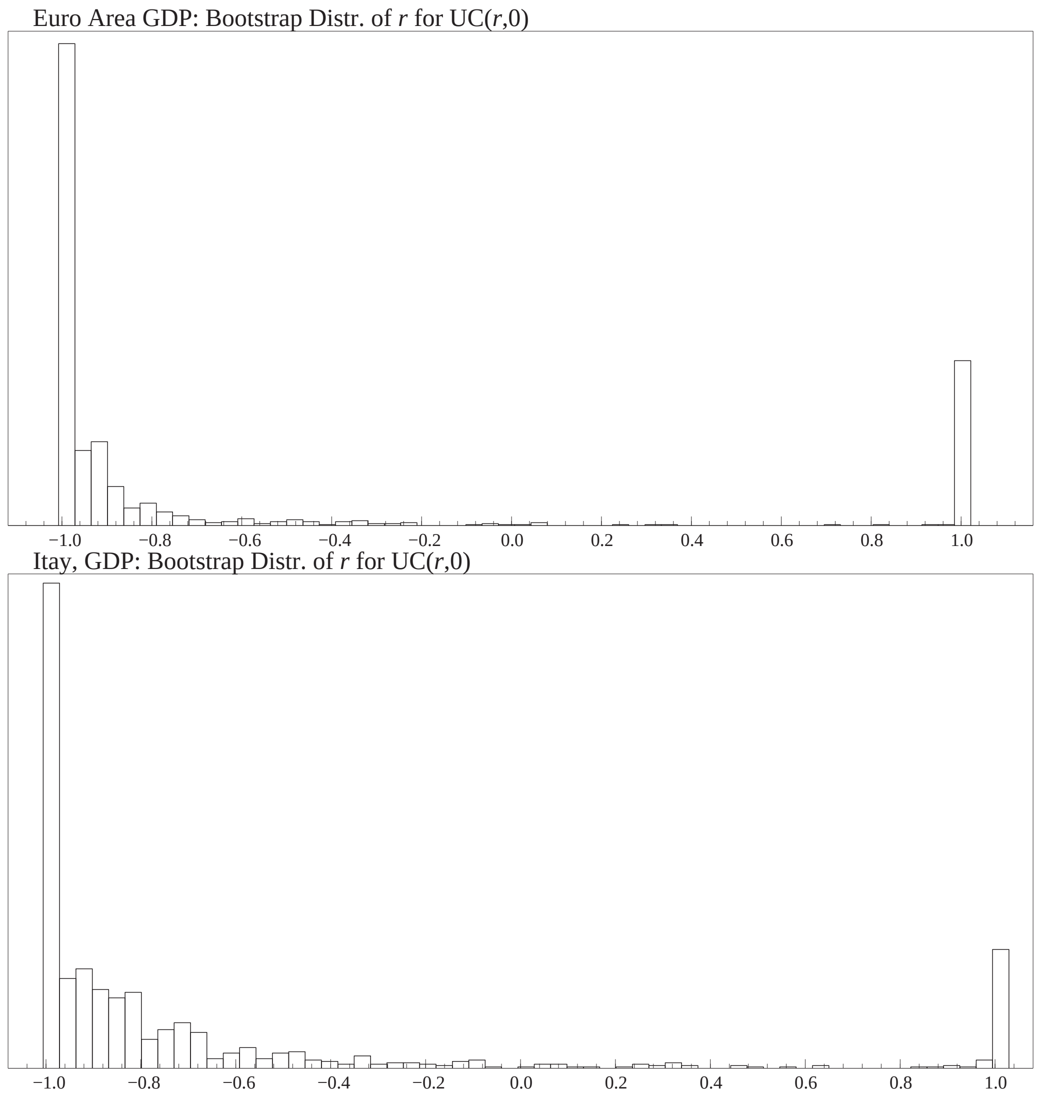

Figure 2: Euro Area and Italian GDP, 1970.1-2002.2. Distribution of the correlation parameter r in 1000 bootstrap samples.

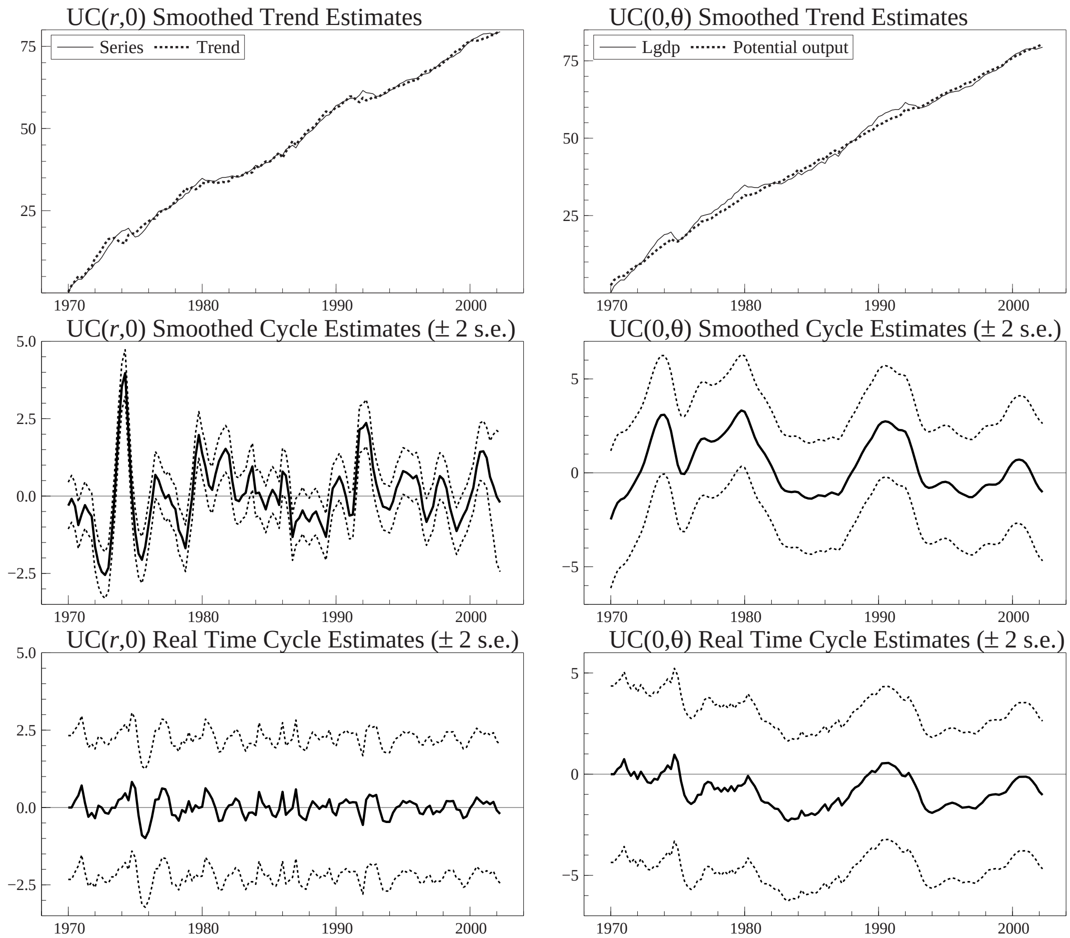

Figure 3: Euro Area GDP, 1970.1-2001.2. Smoothed estimates of trend, smoothed and real time estimates of the cycle arising from the UC(r,0) model (left panels), UC(0,θ) (right panels).

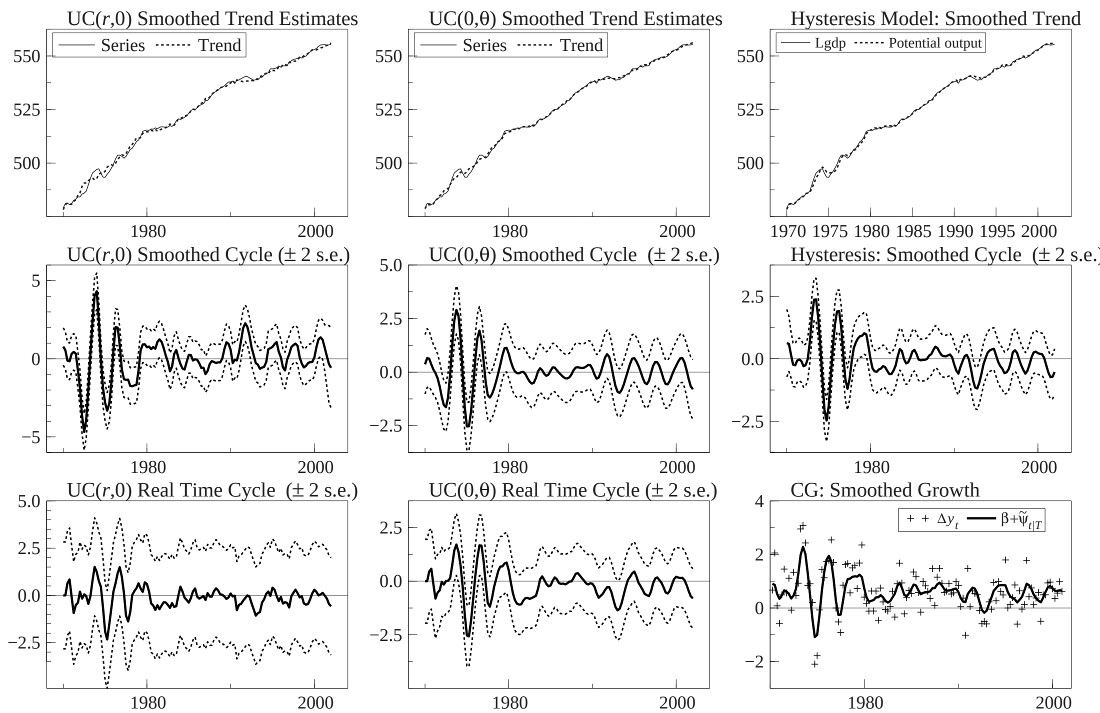

Figure 4: Italian GDP, 1970.1-2001.2. Smoothed estimates of trend, smoothed and real time estimates of the cycle arising from the UC(r,0) model (left panels), UC(0,θ) (centre panels). For the CG-hysteresis model we present the smoothed estimates of the trend (top right panel) and the cycle (middle right panel) and the smoothed estimates of underlying growth (CG model), $\beta + \tilde{\psi}_{t|T}$ (bottom right panel).

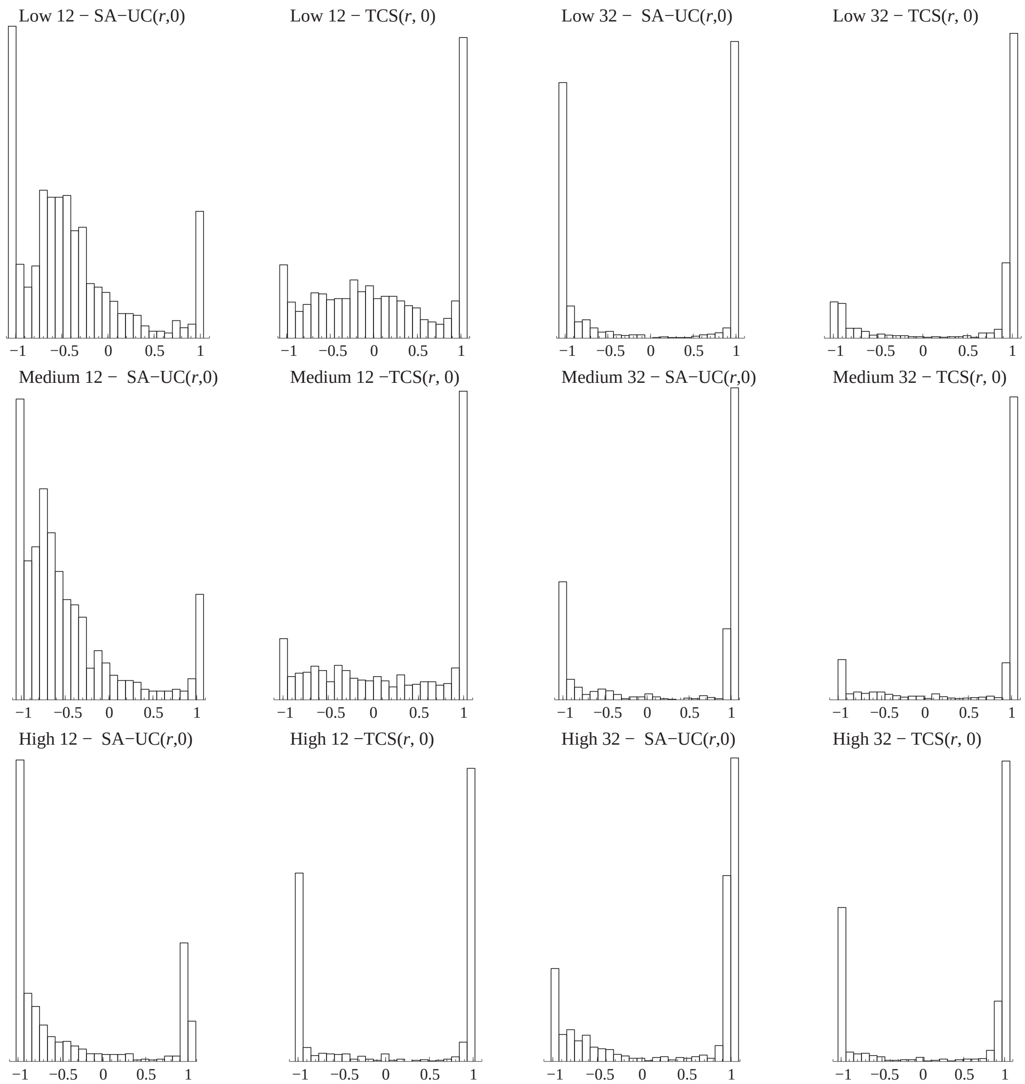

Figure 5: Distribution of the correlation coefficient, r, for the UC(r,0) model estimated on seasonally adjusted data and on the raw simulated data by fitting TCS(r,0). 1000 quarterly series of length $T = 140$ are generated according to orthogonal trend plus cycle plus seasonal models with low, medium and high signal ratios, and cycle periods equal to 12 and 32 quarters.

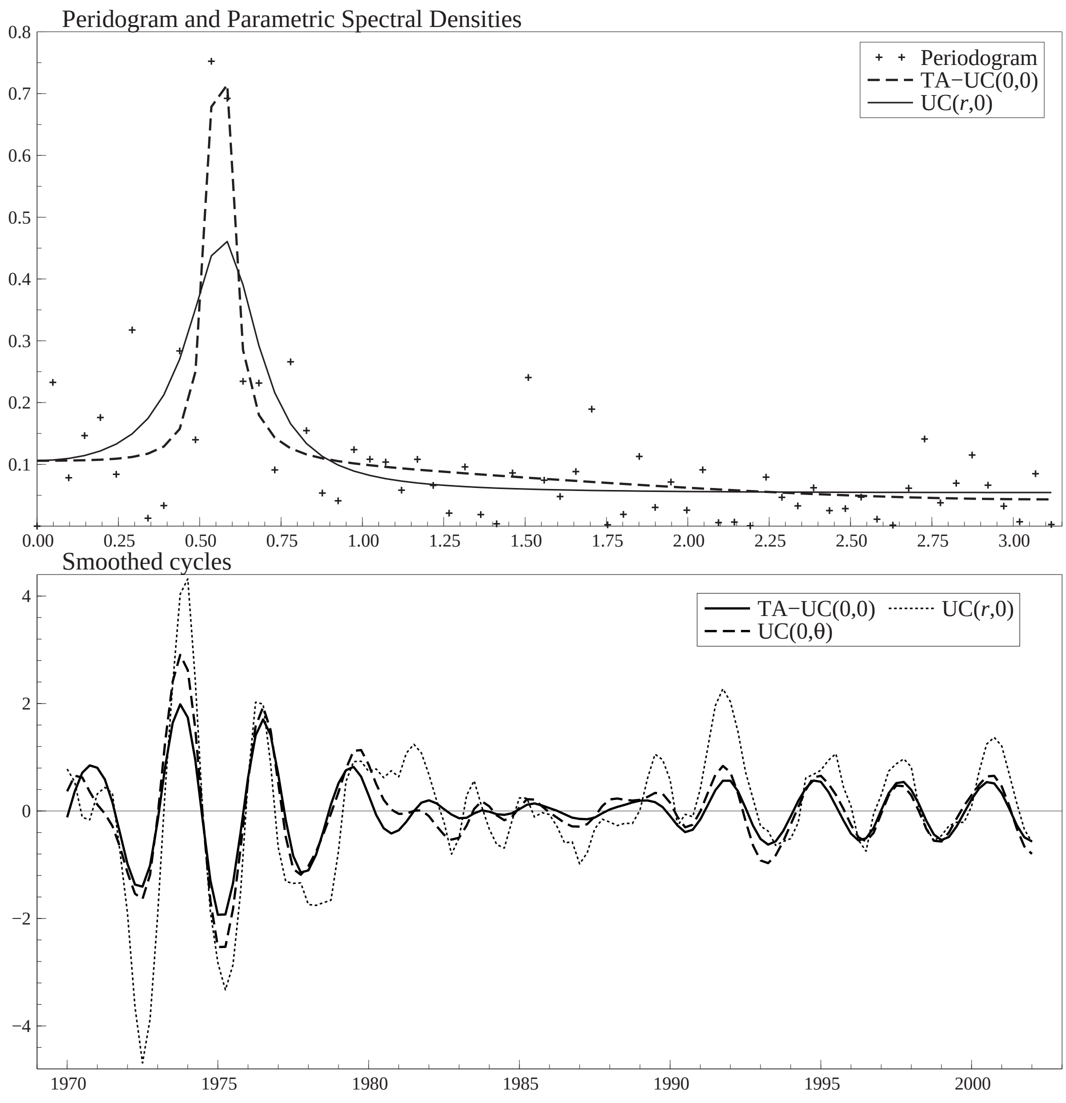

Figure 6: Italian GDP, 1970.1-2002.2. Periodogram, $I(\lambda_j)$, and parametric spectral densities of Δy_t, $g_m(\lambda_j)/(2\pi)$, estimated by the temporally aggregated TA-UC(0,0) model and UC(r,0).

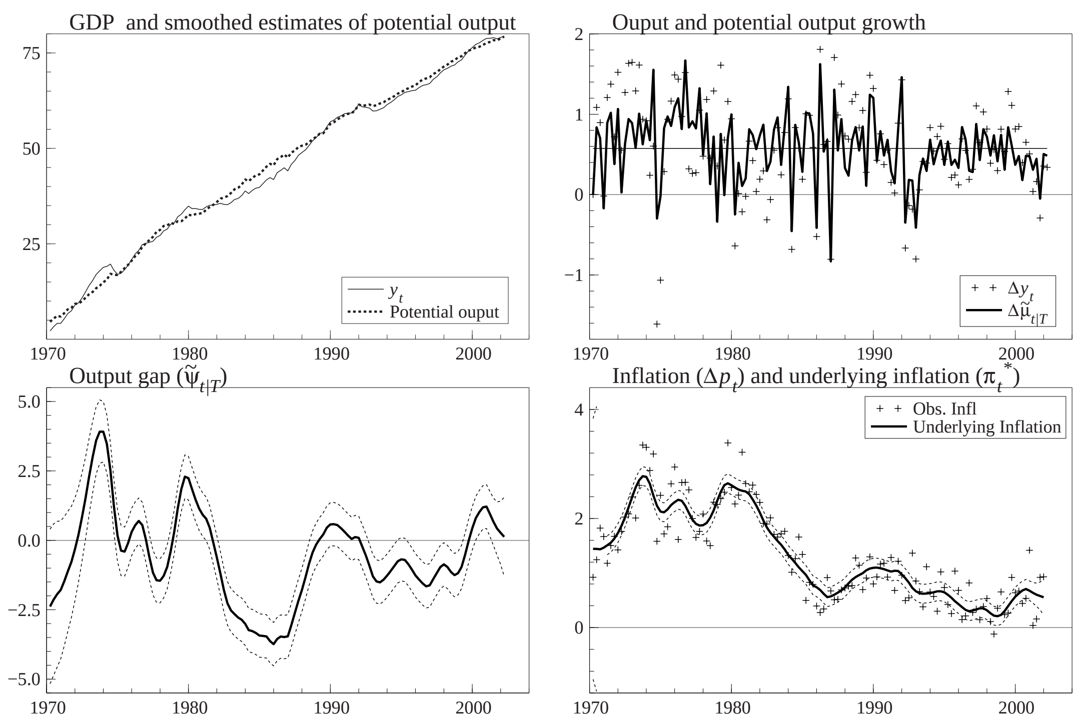

Figure 7: Euro Area GDP and consumer prices (logarithms). Estimates of potential output ($\tilde{\mu}_{t|T}$), potential output growth ($\Delta\tilde{\mu}_{t|T}$), the output gap ($\tilde{\psi}_{t|T}$), and underlying inflation, ($\tilde{\pi}^*_{t|T}$), with 95% confidence intervals, resulting from the bivariate model with correlated disturbances of section 11.